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ON PRODUCT FORM APPROXIMATIONS FOR COMMUNICATION NETWORKS WITH LOSSES: ERROR BOUNDS

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Abstract

This paper studies communication networks with packet or message losses due to collisions, transmission errors and finite buffer constraints. A priori and easily computable error bounds are established for simple product form approximations. These approximations are based on ignoring and/or bounding loss probabilities. Two extreme situations are considered:

(i) Networks with infinite capacities but state dependent loss probabilities.

(ii) Networks with finite capacities (buffers) and losses due to saturated buffers.

The error bounds are of order β, thought of as small, when:

(i) The loss probabilities are uniformly bounded by β, or

(ii) The steady state probability of capacity excess is of order β.

The results provide formal justification for practical engineering and seem of interest for further extension to more complex communication networks.

Key-words

Packet switching * circuit switching * communication networks * loss probabilities * product form approximations * error bounds.

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1 INTRODUCTION

Background

Roughly speaking, a communication or computer network consists of a collection of nodes, representing switching devices, input/output channels, store and forward buffers and processors, which are interconnected by channels to transmit messages or packets between these nodes (direct end-to-end connections in circuit switch systems and from node to node in packet switch networks).

Performance evaluation has become an integral part of present-day computer and communications networking in order to evaluate system efficiency, for optimal design and to assist on line control. Queueing network modeling has become a generally accepted tool in which the communications network is represented by interconnected service stations with transmission or processing times modeled as random services.

Motivation

Unfortunately, closed form expressions for these queueing networks are not generally available when practical phenomena like packet or message losses are involved. Such losses may naturally arise due, for instance, to collisions resulting from time-slotting, limited resource contentions, excess of saturated finite buffers, transmission errors or link failures and usually depend on the workloads at the different stations.

In present-day and even more future communications, though, losses seem to become increasingly less frequent. Switching devices become more advanced, special protocols are devised to avoid collisions, mechanisms for transmission error connections are more commonly installed and, most notably, the capacity of optical transmission fibres is astronomic compared to old-fashioned trunk groups.

As numerical computations or simulations can be most expensive while in absence of losses simple product form expressions might be applicable, it thus seems appealing to simply ignore these losses or to provide uniform bounds for their magnitude, so as to obtain simple approximations. Clearly, such approximations are intuitively justified only when the occurrence of losses is expected to be small.

However, no formal justification, other than by intuitive and numerical support, of such practical approximations seems to be available. In particular, no a priori quantification of the imprecision of such results has been reported.

Results

This paper aims to provide formal justification by analytically establishing a priori and easily computable error bounds on the accuracy of simple product approximations for queueing or communication networks which are based on ignoring and bounding loss probabilities. We study two extreme situations:

1. Communication networks with infinite storage capacities but load dependent loss probabilities such as those due to collisions by time slotting, resource contention, transmission errors or link failures.

2. Communication networks with finite buffers or capacities and losses merely due to saturation of these buffers. Here these buffers must typically be thought of as being large enough to guarantee small loss probabilities.

The error bounds are of order β when

1. The loss probabilities are uniformly bounded by β.
2. The steady state probability of saturations is of order β.

In addition, monotonicity properties are established on the effect of the loss probabilities. The error bound results are intuitively appealing and provide formal practical support for simplifying modeling assumptions in engineering situations to obtain quick estimates of system performance.

Furthermore, the proof technique, as based on Markov reward arguments, is of interest in itself and seems promising for further application to obtain a priori error bounds for computational simplifications of complex communication or random access structures.

Related literature

Recently in [13] a special result was obtained related to (i) for the total delay in a two-stage transmission network. The present paper concerns throughputs and extends to arbitrary multi-stage communication networks. Most notably, though, it also includes losses due to finite buffers which requires special technical details (see Lemma 2).

The proof technique is along the lines of a general perturbation theorem developed in [14]. However, the actual details are not a direct transposition and require verification and extension of technical conditions.

Outline

The organization is as follows. In section 2 we present the two models of interest. Section 3 provides the product form approximations, the main error bound results and a discussion of possible extensions. The technical proofs of these results are given in section 4.

2 MODEL

First we give a general queueing network description in section 2.1. Next, we argue specific communication network applications in section 2.2. Section 2.3 describes the loss mechanisms considered. Section 2.4 illustrates that generally the systems under investigation cannot have a product form.

2.1 GENERAL STRUCTURE

We consider a queueing or communciation network which consists of N service or transmission facilities, hereafter called stations, numbered 1,...,N. Packets or messages to be transmitted through the network, hereafter called jobs, arrive at station i according to a Poisson process with parameter λ_i, $i=1,\dots,N$.

Upon completion of a packet handling or a message transmission, hereafter called service of a job, at station i, a job needs another service at station j with probability p_{ij} or it is completed and leaves the system with probability p_{io}.

The packet rate or transmission speed, that is, the mean number of packets processed or messages transmitted per unit of time, hereafter called service capacity, at station i when n_i jobs are present is $\mu_i(n_i)$. Here, the natural assumption is made that $\mu_i(n_i)$ is nondecreasing in n_i:

(2.1) $$\mu_i(n_i+1) \geq \mu_i(n_i).$$

Further, we adapt the standard approach in queueing network modeling to assume or rather model the underlying structure to be exponential, that is, we assume exponential service times. Though unrealisitc, most notably for packet transmissions, the exponentiality assumption is justifiable for analysis purposes as per various arguments in comment 8 in section 3.3.

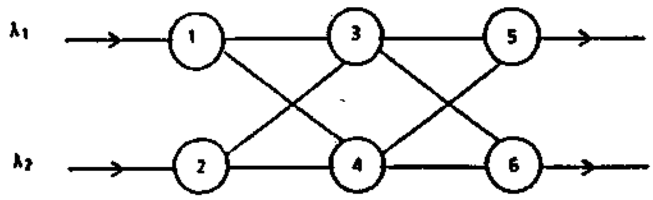
2.2 SPECIAL APPLICATIONS

The above formulation is rather abstract but meant to be so in order to cover various more concrete communication descriptions at the same time. This section merely aims to briefly describe some major standard communication structures that are globally included so as to further motivate the analysis. For those familiar with these structures and their queueing network representations it will probably be redundant. Also, excellent descriptive books as [2], [5], and [9] are referred to for the interested reader.

Roughly, in a communications network messages or packets are to be transported from one end point to another. As a most simple example, direct fixed end-to-end-point connections can be set up (for example, by private lines). However, as the end points of these communications are usually highly variable, a much less costly solution is to make use of intermediate switches. Let us discuss some major switch mechanisms and argue their inclusion in the queueing network setting.

Store and forward switching
In store and forward switching a communnication between two end points is allowed to be asynchronous and to be substantiated in intermediate stages from node to node. Here a node represents a switch or transmitter which hands over a message or packet of information (such as a computer file) to a selected neighbouring node. This message is stored at this node and delayed

for some random time, after which it is forwarded to a next node which is deterministically or randomly selected. Here the nodes can thus be seen as service stations at



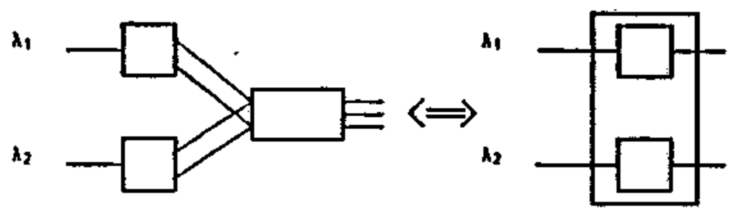
which jobs are delayed or are being served and where the actual transmission from one node to another by interconnecting links takes place in negligible time. In particular, random routing can hereby be involved so as to take into account different end-to-end point directions or dynamic routing depending on nodal loads. This type of switching mostly applies to communications in computer networks (also see below). But it may also be applicable to teletraffic situations when long distance communications are set up in successive stages along various regional (circuit switch) networks (for example, by satellite communications).

Packet switching

Packet switching, which applies most typically to computer (data) communications, works essentially similar to store and forward switching but with messages broken up in individual parts (frames) that are more or less independently transported, possibly taking different routes, on a store and forward basis across the intermediate nodes. The effect of transmission errors is hereby reduced but contention for scarce resources (e.g. single interconnecting links) may increase.

Circuit switching

In circuit switching, which applies more typically to classical teletraffic (voice) communications, a virtual path between two end points along various transmission (e.g. trunk) groups is to be set up, and one link (or one unit of transmission capacity) of each of these groups is occupied at the same time during the total (random) time of the communication.



Circuit switch networks can be modeled in the above described general framework in various ways. Essentially, each possible virtual path should then be seen as a separate parallel station and a communication (job) will require transmission (or service) by only one station. However, due to the possible use of common trunkgroups the parallel service stations, in particular the acceptation or rejection of jobs and losses due to errors, are highly interdependent.

Parallel processing

The modeling of parallel but interdependent service stations also naturally applies to parallel processors or architectures in computer networks. Here, interdependencies may result from commonly used but restricted resources such as a memory disk or drum.

2.3 LOSS DESCRIPTIONS

So far, we have only mentioned in section 2.2 that station interdependencies may be involved. These interdependencies may particularly lead to losses. In what follows we will merely focus on the phenomenon of losses. Further, for convenience of presentation and clarity, we will give a separate presentation for two extreme situations, addressed as models 1 and 2. It is mentioned in advance, though, that combination of the results that will follow is also possible when the two models are combined. By $\bar{n}=(n_1,n_2,\ldots,n_N)$ we denote the number of jobs n_i at station i, $i=1,\ldots,N$.

Model 1 (Infinite capacities; small loss probabilities)

In this case, the stations are assumed to have infinite storage capacities. However, when a job completes service at station i and requests to route to station j, while the system is in state $\bar{n}$, the job is blocked and lost (that is, it leaves the system as an unsuccessful job) with probability

$$\beta_{ij}\,(\bar{n}).$$

When not blocked, it instantaneously routes to station j. Herein, we also allow j=0 to model that a job which leaves the system is unsuccessful (for instance, in the case of a transmission error) so that it is to be considered as lost. Let us briefly argue some possible causes for these losses.

(Collisions)

The actual transmission from one station to another may take place by some single digitized switching device which can process only one job (think of a single packet) per time slot. But as more jobs, possibly from different stations, may complete their service in one and the same time slot, collisions can arise by which jobs (packets) are lost.

(Transmission errors)

The actual transmission or switching of a single packet may lead to an error, say with some state independent probability p. But the actual detection of this error and its consequences, such as delay times for receiving a negative or positive response and successful retransmission or not of the frame in which the error has occurred, will clearly depend on the system loads (cf. [17], Chapter 7).

(Error corrections)

Relatedly, though error correction mechanisms may be implemented, these are usually not 100% covering so that packets and whole messages might still get lost. For example, in the so-called backward error correction mechanism (cf. [17], Chapter 5), a fixed limited number of overhead bits is reserved for errors. However, in highly loaded situations also these bits may have been consumed so that a state dependent incorrectable error and loss may still occur.

The frequencey of the above type of errors, though, or rather, the probability of a loss in an arbitrary state, must typically be thought of as being small, which will motivate theorem 1 in the next section.

Model 2 (Finite buffer excess)

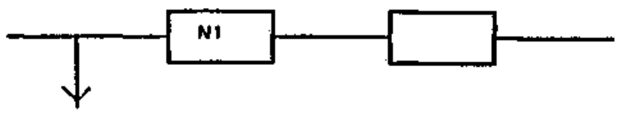
In this case, the stations are considered, as is natural, to have a finite storage capacity or buffer. Say, station i cannot contain more than B_i jobs at a time. When a job enters the system or completes a service at a station i and wishes to enter station j, it is blocked and lost when station j is saturated, that is, when $n_j=N_j$. Otherwise it is accepted. (We thus exclude losses as under model 1.)

This type of blocking and losses is most naturally involved in both packet and circuit switching networks. However, the finite capacities in present-day applications, such as the capacities of optical fibres, become extremely large. One may thus expect the occurrence of a saturation and thus losses to be rare which will motivate theorem 2 in the next section.

2.4 NO CLOSED (PRODUCT) FORM EXPRESSION

Generally, queueing networks with state dependent loss probabilities will not exhibit a closed form expression for the steady state distribution of station loads $\bar{n}$. As no precise reference for this general statement can be provided this section contains two simple illustrations to support this statement and thus to motivate our further analysis.

(i) (A finite tandem structure)
Consider a simple network consisting of two successive service stations and a finite capacity constraint of no more than N_1 jobs at station 1.



As per general analysis in [3] a notion of rate balance per job or per station in any state is necessary and sufficient for a standard type product form result of the form:

$$\pi(n_1,n_2) = c\ g(n_1,n_2) \left(\frac{1}{\mu_1}\right)^{n_1} \left(\frac{1}{\mu_2}\right)^{n_2} .$$

But balance per job or station is directly shown to be violated. For example, let N_1=10 and consider the state (10,2). Then the rate out of that state due to a departure at station 2 is given by

$$\pi(10,2)\ \mu_2(2) > 0.$$

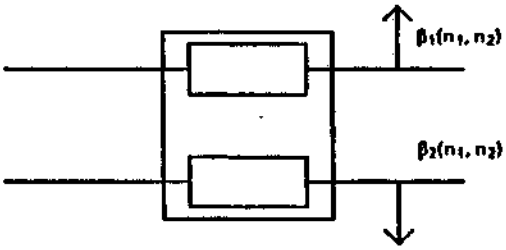
But the rate into that state due to an arrival at station 2 is equal to:

$$\pi(11,1)\ \mu_1(11) = 0,$$

as the state (11,1) is not possible. The rate out and rate into state (10,2) due to station 2 are thus not balanced, so that necessarily, as per this reference, a product form of the above structure fails.

(ii) **(State dependent loss probabilities)**

Consider a network of two parallel service stations and state dependent transmission error loss probabilities $\beta_1(n_1,n_2)$ for station 1 and $\beta_2(n_1,n_2)$ for station 2.



As per [3] or [4] one then concludes that a product form of the above form applies if and only if reversibility (cf. [4]) is guaranteed, which requires for example for any state (n_1,n_2):

$$\beta_1(n_1-1,n_2-1)\ \beta_2(n_1,n_2-1) =$$

$$\beta_2(n_1-1,n_2-1)\ \beta_1(n_1-1,n_2).$$

This is satisfied only for very special situations. With $\beta_1(.,.)=\beta_2(.,.)$, for example, it directly leads to the requirement that the error loss probabilities may depend on (n_1,n_2) only by their sum n_1+n_2. For example, it would fail if $\beta_1(.,.)=\beta_2(.,.)=f(n_2)$ for any non-constant function f.

3 PRODUCT FORM APPROXIMATIONS AND ERROR BOUND RESULTS

3.1 PRODUCT FORM COMPARISON MODEL

As the systems of interest as described in section 2 do not generally exhibit a closed product form expression, so that highly expensive numerical or approximative computations are to be employed for evaluation purposes, the following purely artificial modified model, called model 3, is proposed for computational convenience.

Model 3 (Fixed loss probability α)

This model is identical to model 1, that is with infinite station capacities but possible loss probabilities, for example due to transmission errors. In contrast, however, these loss probabilities are assumed to be state independent, that is, for some value α and all i,j (j=0 included) and all states $\bar{n}$:

$$\beta_{ij}(\bar{n}) = \alpha.$$

We will compare this system with model 1 using both a value $\alpha>0$ and $\alpha=0$, while when comparing this system with model 2 we will only use the value $\alpha=0$. In the latter case, furthermore, that is, when compared with system 2, we extrapolate the service capacities beyond values B_i by $\mu_i(n_i)=\mu_i(B_i)$ for all $n_i \geq B_i$, where we assume that the system remains ergodic.

Product form

For arbitrary α and with c a normalizing constant, the steady state distribution of model 3 exhibits the product form:

$$(3.1) \qquad \pi_{\alpha}^{3}(\bar{n}) = c \prod_i [\gamma_i]^{n_i} \left[\prod_{k=1}^{n_i} \mu_i(k)\right]^{-1},$$

where $\{\gamma_i\}_{i=1}^{N}$ is assumed to be the unique solution of the traffic equations:

$$(3.2) \qquad \gamma_j = \lambda_j + \sum_i \gamma_i \, p_{ij} \, [1-\alpha] \qquad (j=1,\ldots,N).$$

3.2 APPROXIMATIONS AND MAIN RESULTS

We will be interested in the system throughput, that is the mean number of successful system departures per unit of time not due to losses. Let θ^1, θ^2 and θ_{α}^{3} denote these throughputs for model 1, 2 and 3 with value α, respectively. While θ^1 and θ^2 have no general easily computable form, the value θ_{α}^{3} can be concluded rather standardly as based on the product form expression (3.1) by

$$(3.3) \qquad \theta_{\alpha}^{3} = \sum_{\bar{n}} \pi_{\alpha}^{3}(\bar{n}) \sum_j \mu_j(n_j) \, p_{j0}[1-\alpha] = [1-\alpha] \sum_j \gamma_j \, p_{j0}.$$

The following two theorems will compare the throughputs θ^1 and θ^2 with their approximate values θ^3_α. In the first place, these theorems formally prove that uniformly estimating loss probabilities from below or above will lead to upper or lower throughput bounds. This may seem intuitively obvious but, as per remark 1 in section 3.3 and also referring to the claimed validity of the bounds in the non-exponential case as per remark 5 in section 3.3, it is not trivial. In the second place, and most importantly, the theorems provide a priori error bounds which can be computed directly. Theorem 1 extends a special result in [13] for two-stage service structures to multiple service-stage networks. Theorem 2 appears to be totally new. Roughly speaking, it expresses the effect of finite buffers in the steady state probability of a solvable infinite system to exceed the buffer limits. For presentational convenience, let

$$\Gamma = \sum_i \gamma_i$$

$$\lambda = \sum_i \lambda_i = \theta^3_0$$

$$\Phi(\bar{n}) = \sum_i \mu_i(n_i)$$

$$B = \bigcup_i \{\bar{n} \mid n_i \geq B_i\}$$

Theorem 1 *(Monotonicity results and error bounds of product form approximations with constant loss probabilities for the transmission error loss model 1).* Assume that for all $\bar{n}$ and i,j and some value β

(3.4) $$\beta_{ij}(\bar{n}) \leq \beta.$$

Then

(3.5) $$\theta^3_\beta \leq \theta^1 \leq \theta^3_0$$

(3.6) $$0 \leq \theta^3_\beta - \theta^3_0 \leq \beta\,\Gamma$$

Theorem 2 *(Error bound for infinite product form approximations for the finite buffer loss model 2)* With $\pi_0^3(.)$ given by the product form expression (3.1) and (3.2) with $\alpha=0$, we have:

(3.7) $$0 \le \theta_0^3 - \theta^2 \le \lambda\ \pi_0^3(B) + \sum_{\bar{n}\in B} \pi_0^3(\bar{n})\ \Phi(\bar{n})\ .$$

The proofs of these theorems will be provided in section 4. First, in section 3.3 below, we will briefly discuss the results and argue some possible extensions.

3.3 DISCUSSION AND EXTENSIONS

Here we aim to briefly argue that the results obtained are non-trivial but intuitively supporting for practical engineering. Furthermore, several extensions will be mentioned without proofs.

1 **(Monotonicity results)** The monotonicity results in (3.4) and (3.7) may seem trivial. However, by blocking more messages at one place, the loss rates at other places may be enlarged (note, for example, that no conditions are imposed on $\beta_{ij}(\bar{n})$), so that the actual impact is unpredictable. In fact, one can give counterintuitive examples in which the throughput for a particular stochastic realization in a system is enlarged by rejecting more jobs (e.g. [1],[11]). This counterintuitive phenomenon becomes even more puzzling when the product form approximations are claimed to be valid for arbitrary and thus also deterministic services, as under 5 below in various situations.

2 **(Error bound 3.6)** The error bound in (3.6) is intuitively supported as a mean number of λ jobs arrive at the system per unit of time while each of these jobs will at least once experience a loss with probability β.

3 **(Error bound 3.7)** The error bound in (3.7) also seems intuitively appealing as it represents the mean number of arrivals and service completions per unit of time in steady state when at least one of the stations is saturated.

Clearly, for large buffer sizes B_i this error bound will be rather small and of the order of the steady state probability $\pi_0^3(B)$.

4 (Other measures) Similar results can be obtained also for other performance measures such as the total number of jobs at the system, the sojourn time of a job in the system or the mean workload of a particular station. As the technical details will be quite similar, we leave such extensions to the interested reader.

5 (Combination of models) As stated earlier, the results can easily be combined if we want to consider a system with losses due to both finite buffers and (transmission) error probabilities $\beta_{ij}(\bar{n})$. For presentational clarity, however, we preferred to analyze both loss features separately.

6 (More accurate approximations for model 1) In fact, the product form approximations for model 1 can be somewhat improved by setting

$$\gamma_{ij} = \inf_{\bar{n}} \beta_{ij}(\bar{n})$$

$$\delta_{ij} = \sup_{\bar{n}} \beta_{ij}(\bar{n})$$

and using product form approximations as per (3.1) and (3.2) with α=0 and

p_{ij} replaced by $p_{ij}[1-\gamma_{ij}]$ to obtain a throughput upper bound and
p_{ij} replaced by $p_{ij}[1-\delta_{ij}]$ to obtain a throughput lower bound.

However, with $\beta=\max_{ij}\delta_{ij}$, the proof technique followed would still lead to an error bound $\beta\lambda$.

7 (Arrival losses included in model 1) Also state dependent arrival losses $\beta_{0j}(\bar{n})$ could have been included in model 1. This, however, would have required some slightly more technical details and would have led to an error bound

$$\beta\Gamma + \lambda \sup_{\bar{n}} \beta_{oj}(\bar{n}).$$

As transmission errors within the network were considered as the main interest, we have excluded this extension for presentational convenience.

8 (Non exponential services) Exponentiality assumptions are often far from realistic. Most notably, the transmission of packets, such as of one packet per discrete-time slot, typically requires a simple discrete or even deterministic distribution per message processing. However, the assumptions of exponential distributions seem justifiable for various reasons.

(i) (Flow equations) Formally, the exponential assumptions are needed to justify an analysis by Markov chain transition equations. However, in engineering situations the same equations are applied simply as flow equations, representing the mean number of transitions or service completions per unit time, regardless of specifying underlying randomness.

(ii) (Insensitivity properties) Performance measures like a throughput are generally quite robust (or insensitive) for service distributional forms and are primarily determined by their means. In particular, the product form approximations presented are 100% insensitive, provided the service disciplines are of special form, such as of processor sharing or infinite server type. This suggests that in those cases also the bounds will be 100% insensitive as bounds (that is, be valid for arbitrary service distributions with the same mean service rates). Indeed, formal but technical proofs of such statements can be provided (cf. [12]).

(iii) (Discrete-time analysis) Particularly in the case of packet transmissions, a discrete-time analysis with transmissions during exactly an integer number of time-slots (possibly just one), would have been appropriate. This would lead to complications of multiple changes at the same time. Nevertheless, the same product form results could then have been concluded, given the mean service rates $\mu_1(n_1)$ per unit time, as per recent product form results for discrete-time queueing networks (e.g. [6]).

4 PROOFS

4.1 NOTATION AND DISCRETE-TIME TRANSFORMATION

In order to compare the three models in an inductive manner, we will first transform the continuous-time descriptions into discrete-time descriptions. To this end, we assume that

(4.1) $$Q = [\ \Sigma_i \lambda_i + \sup_{\bar{n}} \Sigma_i \mu_i(\bar{n})] < \infty$$

and we make the convention to denote expressions for model 1, 2 and 3 with a superscript 1, 2 and 3.

Now, let P^1, P^2, P^3 be the one-step transition matrices at S^1, S^2, S^3 as according to the uniformization method (eg. [10], p.110) for model 1, 2, 3, respectively. More precisely, for a state $\bar{n}$, let $\bar{n}+e_i$ or $\bar{n}-e_i$ denote the same state with one job more, respectively less, at station i, where $\bar{n}-e_0=\bar{n}$, and define $\bar{n}-e_i+e_j$ as $(\bar{n}-e_i)+e_j$.

Then,

$$\begin{cases} p^1(\bar{n},\bar{n}+e_i) = \lambda_i Q^{-1} & (i\le N) \\ p^1(\bar{n},\bar{n}-e_i+e_j) = \mu_i(n_i)p_{ij}\ [1-\beta_{ij}(\bar{n})]\ Q^{-1} & (i,j\le N) \\ p^1(\bar{n},\bar{n}-e_i) = \mu_i(n_i)\left[1 - \Sigma_{j=1}^N\ p_{ij}\ [1-\beta_{ij}(\bar{n})]\right]\ Q^{-1} & (i\le N) \end{cases}$$

$$\begin{cases} p^2(\bar{n},\bar{n}+e_i) = \lambda_i\ 1_{(n_i<B_i)} Q^{-1} & (i\le N) \\ p^2(\bar{n},\bar{n}-e_i+e_j) = \mu_i(n_i)p_{ij}\ 1_{(n_j<B_j)} Q^{-1} & (i,j\le N) \\ p^2(\bar{n},\bar{n}-e_i) = \mu_i(n_i)\left[1 - \Sigma_{j=1}^N\ p_{i,j}\ 1_{(n_j<B_j)}\right] Q^{-1} & (i\le N) \end{cases}$$

$$\begin{cases} p^3(\bar{n},\bar{n}+e_i) = \lambda_i\ Q^{-1} & (i\le N) \\ p^3(\bar{n},\bar{n}-e_i+e_j) = \mu_i(n_i)p_{ij}[1-\alpha]\ Q^{-1} & (i,j\le N) \\ p^3(\bar{n},\bar{n}-e_i) = \mu_i(n_i)\ \left[1 - [1-\alpha]\Sigma_{j=1}^N p_{ij}\right]\ Q^{-1} & (i\le N) \end{cases}$$

and

$$p^s(\bar{n},\bar{n}) = [\ 1 - \Sigma_{ij}\ \ p(\bar{n},\bar{n}-e_i+e_j)] \qquad (s=1,2,3 \text{ and all } \bar{n})$$

Define operators T^1, T^2, T^3 and for all t=0,1,2,...: operators T^1_t, T^2_t, T^3_t on real-valued functions $f: S^s \to \mathbb{R}$ by

$$(4.3)\qquad \begin{cases} T^s f(\bar{n}) = \Sigma^N_{i,j=0}\ p^s(\bar{n},\bar{n}-e_i+e_j)\ f(\bar{n}-e_i+e_j) \\ T^s_{t+1} f = T^s(T^s_t f) \qquad (t\geq 0) \\ T^s_0 = f, \end{cases}$$

for s = 1,2,3, and with

$$(4.4)\qquad \begin{cases} r^1(\bar{n}) = \Sigma_i \mu_i(n_i) p_{i0} [1-\beta_{i0}(\bar{n})]\ Q^{-1} \\ r^2(\bar{n}) = \Sigma_i \mu_i(n_i) p_{i0}\ Q^{-1} \\ r^3(\bar{n}) = \Sigma_i \mu_i(n_i) p_{i0} [1-\alpha]\ Q^{-1} \end{cases}$$

let the functions V^1_t, V^2_t, V^3_t, t=0,1,2,... be given by:

$$(4.5)\qquad V^s_t(\bar{n}) = \Sigma^{t-1}_{k=0}\ T^s_k\ r^s(\bar{n}) \qquad (\bar{n}\in S^s)\ (s=1,2,3)$$

Verbally, $V^s_t(\bar{n})$ represents the total expected reward over t periods of the Markov reward chain with one-step transition probabilities P^s and one-step reward $r(\bar{m})$ per step whenever the system is in the state $\bar{m}$, given that the chain starts in state $\bar{n}$ at time (step) 0. Then, by standard Markov reward arguments (e.g. [8], [10]) and the method of uniformization (e.g. [10], p.110) we can conclude:

$$(4.6)\qquad \theta^s = \lim_{t\to\infty} \frac{Q}{t}\ V^s_t(\bar{n})$$

for arbitrary initial state $\bar{n}\in S^s$ and s=1,2,3. Consequently, the throughputs θ^s can be compared by comparing the expressions (4.5). This in turn, can be achieved inductively by virtue of the one-step or dynamic reward relation; directly following from (4.3) and (4.5):

$$(4.7)\qquad V^s_{t+1} = r^s + T^s\ V^s_t \qquad (t\geq 0,\ s=1,2,3).$$

4.2 PROOF OF THE THEOREM 3.1

4.2.1 Proof of (3.5).

First, let us compare model 1 where we have infinite buffers but state dependent (transmission) error loss probabilities $\beta_{ij}(\bar{n})$ and model 3 where $\alpha=\beta$ (that is, with infinite buffers but with majorizing state independent fixed loss probabilities β). By (4.7) and the fact that $S^1 = S^3$ we can write:

$$(4.8) \qquad (V_t^1 - V_t^3)(\bar{n}) =$$

$$(r^1 - r^3)(\bar{n}) + T^1V_{t-1}^1(\bar{n}) - T^3V_{t-1}^3(\bar{n}) =$$

$$(r^1 - r^3)(\bar{n}) + (T^1 - T^3)V_{t-1}^3(\bar{n}) + T^1(V_{t-1}^1 - V_{t-1}^3)(\bar{n}).$$

From (4.4) we obtain

$$(4.9) \qquad (r^1 - r^3)(\bar{n}) =$$

$$\Sigma_i \mu_i(n_i) p_{i0}\, Q^{-1}[1 - \beta_{i0}(\bar{n})] -$$

$$\Sigma_i \mu_i(n_i) p_{i0}\, Q^{-1}[1 - \beta] = \Sigma_i\, \mu_i(n_i) p_{i0} Q^{-1}[\beta - \beta_{i0}(\bar{n})].$$

By comparing the transition probabilities (4.2) for model 1 and model 3 with $\alpha=\beta$ we similarly obtain for arbitrary k and $\bar{n}$:

$$(4.10) \qquad (T^1 - T^3)V_k^3(\bar{n}) =$$

$$\Sigma_i\, \mu_i(n_i)\, Q^{-1}\, \Sigma_{j=1}^N\, p_{ij}\, [\beta - \beta_{ij}(\bar{n})]\, [V_k^3(\bar{n}-e_i+e_j) - V_k^3(\bar{n}-e_i)].$$

Substituting (4.9) and (4.10) in (4.8) and using the lower estimate 0 from (4.28) of lemma 1 below, we conclude for arbitrary $\bar{n}$:

$$(4.11) \qquad (V_t^1 - V_t^3)(\bar{n}) \geq$$

$$T^1\,(V_{t-1}^1 - V_{t-1}^3)(\bar{n}) =$$

$$\sum_{\bar{n}'} P^1(\bar{n},\bar{n}')\ (V^1_{t-1}-V^3_{t-1})(\bar{n}') \geq \ldots$$

$$\sum_{\bar{n}'} (P^1)^k(\bar{n},\bar{n}')\ (V^1_0-V^3_0)(\bar{n}') = 0 \qquad\qquad (t\geq 0)$$

where $(P^1)^k$ denotes the k-th power of the one-step transition matrix (P^1). By letting $t\to\infty$, substituting $\bar{n}=0$ and applying (4.6) we then obtain

$$(4.12)\qquad \theta^1 = \lim_{t\to\infty} \frac{Q}{t}\, V^1(\bar{0}) \geq \lim_{t\to\infty} \frac{Q}{t}\, V^3(\bar{0}) = \theta^3 \qquad (\text{for } \alpha=\beta).$$

In a completely similar manner we can also prove $\theta^1\leq\theta^3$ for $\alpha=0$.
(In *fact, this also directly follows* by substituting $\beta=0$ and replacing $\geq$ by $\leq$ signs.) □

4.2.2 Proof of (3.6)

We need to compare model 3 with $\alpha=0$ and $\alpha=\beta$. However, as model 3 with $\alpha=0$ coincides with model 1 with $\beta_{ij}(\bar{n})=0$ for all i,j and $\bar{n}$, we can directly apply the results from section 4.2.1 above with model 1 representing model 3 with $\alpha=0$. The inequality $\theta^3_0 \geq \theta^3_\beta$ is hereby directly proven. Further, by iterating (4.8) and noting that $V^1_0(.) = V^3_0(.) = 0$, we derive for arbitrary $\bar{n}\in S^1$:

$$(4.13)\qquad (V^1_t - V^3_t)\ (\bar{n}) =$$

$$\sum_{k=0}^{t-1} T^1_k \left([r^1-r^3] + [T^1-T^3]\ V^3_{t-k-1}\right)(\bar{n}).$$

From (4.9) and (4.10) with $\beta_{ij}(\bar{n})=0$ for all i,j and $\bar{n}$, and the upper estimate 1 from (4.28) in Lemma 1 below, we also establish:

$$(4.14)\qquad \left|(r^3_0-r^3_\beta)\ (\bar{n}) + [T^3_0-T^3_\beta]\ V^3_k\ (\bar{n})\right| = \left|\sum_i \mu_i(n_i)\ p_{i0}\ [\beta-0]\ Q^{-1} + \right.$$

$$\left.\sum_i \mu_i(n_i)\ Q^{-1} \sum_{j=1}^N p_{ij}\ [\beta-0]\ [V^3_k(\bar{n}-e_i+e_j) - V^3_k(\bar{n}-e_i)]\right| \leq$$

$$\beta\ Q^{-1} \sum_i \mu_i(n_i) \left[\sum_{j=1}^N p_{ij} + p_{i0}\right] \leq \beta\ Q^{-1} \sum_i \mu_i(n_i).$$

Thus, by substituting $\Phi_2(\bar{n}) = \sum_i \mu_i(n_i)$ and combining (4.13) and (4.14), we get:

$$(4.15) \qquad |(V_t^1-V_t^3)(\bar{n})| \le \beta\ Q^{-1} \sum_{k=0}^{t-1} T_k^1\ \Phi_2^1(\bar{n}).$$

Substituting $\bar{n}=\bar{0}$, recalling that $T_k^1=T_k^3$ for model 3 with $\alpha=0$, applying (4.34) from Lemma 2 below with z=2 and noting that

$$(4.16) \qquad \sum_{\bar{n}} \pi_0^3(\bar{n}) \sum_i \mu_i(n_i) = \sum_i \gamma_i = \Gamma.$$

we obtain from (4.15):

$$(4.17) \qquad |(V_t^1-V_t^3)(\bar{0})| \le \beta\ Q^{-1} t\ \Gamma,$$

so that by (4.6):

$$|\theta^1 - \theta^3| \le \beta\ \Gamma.$$

As θ^1 stands for model 3 with $\alpha=0$ and θ^3 for model 3 with $\alpha=\beta$, this proves (3.6). □

4.3 PROOF OF THEOREM 3.2

We need to compare model 2 (with finite buffer sizes B_i but no transmission error losses) with model 3 where $\alpha=0$ (with infinite buffers and also no transmission losses as $\alpha=0$).

Here, the inequality $\theta^2 \le \theta_0^3 = \lambda$ is trivial as in model 3 with $\alpha=0$, no jobs at all will ever be lost.

To prove the error bound from (3.7), first note that $S^2 \subset S^3$, and that $p^2(\bar{n},\bar{n}')=0$ for any $\bar{n}\in S^2$ and $\bar{n}'\notin S^2$. As a consequence, for any k the function $(T^3-T^2)V_k^3$ is well-defined at S^2 by:

$$(T^3-T^2)\ V_k^3(\bar{n}) =$$

$$\sum_{\bar{n}\in S^3} [p^3(\bar{n},\bar{n}') - p^2(\bar{n},\bar{n}')]\ V_k^3(\bar{n}') \qquad (\bar{n} \in S^2).$$

Therefore, similiarly to (4.8), we can write:

$$(4.19) \qquad (V_t^3-V_t^2)(\bar{n}) = (r^3-r^2)(\bar{n}) + (T^3-T^2)\ V_{t-1}^3(\bar{n}) + T^2(V_{t-1}^3-V_{t-1}^2)(\bar{n}).$$

By (4.4) with α=0 for model 3, we directly have $(r^2-r^3)(\bar{n}) = 0$.
Furthermore, observe that $T_k^2 f$ is well-defined at S^2 for any function $f: S^2 \to \mathbb{R}$ by

$$(4.20) \qquad T_k^2 f(\bar{n}) = \sum_{\bar{n}' \in S^2} (P^2)^k (\bar{n},\bar{n}') f(\bar{n}'),$$

where $(P^2)^k$ is the k-th power of the one-step transition matrix (P^2). Similarly to (4.13), we are thus allowed to iterate (4.19) so as to obtain:

$$(4.21) \qquad (V_t^3-V_t^2)(\bar{n}) = \sum_{k=0}^{t-1} T_k^2 \left([T^3-T^2]\, V_{t-k-1}^3\right)(\bar{n}),$$

for any $\bar{n}\in S^2$, where we have used also the fact that $(V_0^3-V_0^3)(.) = 0$. By comparing the transition probabilities (4.2) for models 2 and 3 with α=0, we find for arbitrary k:

$$(4.22) \qquad (T^3-T^2)\, V_k^3(\bar{n}) =$$

$$\sum_i \lambda_i\, Q^{-1}\, 1_{(n_i=B_i)}\, [V_k(\bar{n}+e_i) - V_k(\bar{n})] +$$

$$\sum_i \mu_i(n_i)\, Q^{-1} \sum_j p_{ij}\, 1_{(n_j=B_j)} \left[V_k^3(\bar{n}-e_i+e_j) - V_k^3(\bar{n}-e_i)\right].$$

Also noting that

$$(4.23) \qquad 1_{(\bar{n}\in B)} = 1_{\cup(n_i \geq B_i)}$$

and using the upper estimate 1 from (4.28) of Lemma 1 below, we conclude from (4.22) for any $\bar{n}\in S^2$:

$$(4.24) \qquad \left|(T^3-T^2)\, V_k^3\, (\bar{n})\right| \leq$$

$$\sum_i \lambda_i\, 1_{(n_i=B_i)}(\bar{n}) + \sum_i \mu_i(n_i) \sum_j p_{ij}\, 1_{(n_j=B_j)} \leq$$

$$1_{\cup(n_i=B_i)} \left[\sum_i \lambda_i + \sum_i \mu_i(n_i)\right] \leq 1_{(\bar{n}\in B)} \sum_i \lambda_i + 1_{(\bar{n}\in B)} \sum_i \mu_i(n_i).$$

Noting that T_k^2 is a positive operator for any k (compare (4.11) and (4.20)) (that is, $T_k^2 f \le T_k^2 g$ for any $f \le g$ componentwise) we now conclude from (4.21) and (4.24):

$$(4.25) \qquad |(V_t^3 - V_t^2)(\bar{n})| \le Q^{-1} \textstyle\sum_{k=0}^{t-1} T_k^2 \psi(\bar{n}),$$

where the function $\psi(.)$ is defined by:

$$(4.26) \qquad \psi(\bar{n}) = 1_{(\bar{n} \in B)} \left[\textstyle\sum_i \lambda_i + \sum_i \mu_i(n_i) \right].$$

By substituting $\bar{n}=\bar{0}$ in (4.25) and applying the inequalities (4.33) and (4.34) with z=1 and z=3 from Lemma 2 below, we then derive:

$$(4.27) \qquad |(V_t^3 - V_t^2)(\bar{0})| \le t\, Q^{-1} \sum_{\bar{n} \in B} \pi_0^3(\bar{n}) \left[\lambda + \textstyle\sum_i \mu_i(n_i) \right].$$

Applying (4.6) with $\bar{n}=\bar{0}$, completes the proof of the error bound in (3.7). □

4.4 TECHNICAL LEMMAS

In this section we prove the two technical but essential lemmas that have been used to prove Theorems 1 and 2 (Lemma 1 was applied for both Theorem 1 and 2, while Lemma 2 was necessary to establish the error bound in Theorem 2).

Lemma 1 For model 3 with arbitrary $\alpha>0$, all $\bar{n}$ and j and all $t \ge 0$:

$$(4.28) \qquad \boxed{0 \le V_t^3(\bar{n}+e_j) - V_t^3(\bar{n}) \le 1}$$

Proof

We apply induction on t. For t=0, (4.28) is trivially satisfied as $V_0^3(.)=0$. Suppose that (4.28) holds for all $t \le m$ and all $\bar{n}, j$. The following relations will then be obtained from the dynamic one-step transition equation (4.7). Herein we suppress the superscript 3 and substitute $Q^{-1}=h$.

From (4.7) where we need to substitute the probabilities for P^3 and by decomposing $\mu_j(n_j+1)$ in $\mu_j(n_j+1) = \mu_j(n_j) + [\mu_j(n_j+1) - \mu_j(n_j)]$, we derive for t=m+1 in state $\bar{n}+e_j$:

(4.29) $V_{m+1}(\bar{n}+e_j)$

$$=$$

$$h \sum_i \mu_i(n_i)\, p_{i0}\, [1-\alpha] + h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, p_{i0}\, [1-\alpha] +$$

$$h \sum_i \lambda_i V_m(\bar{n}+e_j+e_i) +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)}\, [1-\alpha] \sum_{\ell=0}^{N} p_{i\ell}\, V_m\, (\bar{n}+e_j-e_i+e_\ell) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha] \sum_{\ell=1}^{N} p_{j\ell}\, V_m\, (\bar{n}+e_\ell) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha]\, p_{j0}\, V_m(\bar{n}) +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)}\, [\alpha]\, V_m(\bar{n}+e_j-e_i) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [\alpha]\, V_m(\bar{n}) +$$

$$\left[1 - h \sum_i \lambda_i - h \sum_i \mu_i(n_i) 1_{(n_i>0)} - h\, [\mu_i(n_j+1) - \mu_j(n_j)]\right] V_m(\bar{n}+e_j).$$

Similarly, by (4.7) again and by artificially adding and substracting terms with coefficient $[\mu_j(n_j+1)-\mu_j(n_j)]$, we obtain for t=m+1 in state $\bar{n}$:

(4.30) $V_m(\bar{n})$

$$=$$

$$h \sum_i \mu_i(n_i) p_{i0}\, [1-\alpha] + h \sum_i \lambda_i\, V_m(\bar{n}+e_i) +$$

$$h \sum_i \mu_i(n_i) 1_{(n_i>0)} [1-\alpha] \sum_{\ell=0}^{N} p_{i\ell}\, V_m(\bar{n}-e_i+e_\ell) +$$

$$h \sum_i \mu_i(n_i) 1_{(n_i>0)} [\alpha] V_m(\bar{n}-e_i) + \left[1-h \sum_i \lambda_i - h \sum_i \mu_i(n_i) 1_{(n_i>0)}\right] V_m(\bar{n})$$

$$=$$

$$h \sum_i \mu_i(n_i)\, p_{i0}\, [1-\alpha] + h \sum_i \lambda_i\, V_m(\bar{n}+e_i) +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)}\, [1-\alpha] \sum_{\ell=0}^{N} p_{i\ell}\, V_m(\bar{n}-e_i+e_\ell) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha] \sum_{\ell=1}^{N} p_{j\ell}\, V_m(\bar{n}) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha]\, p_{j0}\, V_m(\bar{n}) +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)}\, [\alpha]\, V_m(\bar{n}-e_i) +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [\alpha]\, V_m(\bar{n}) +$$

$$\left[1 - h \sum_i \lambda_i - h \sum_i \mu_i(n_i) 1_{(n_i>0)} - h\, [\mu_j(n_j+1)-\mu_j(n_j)]\right] V_m(\bar{n}).$$

By substracting (4.30) from (4.29), where we keep in a fifth and seventh term that are actually equal to 0 for clarity and arguments below, we obtain:

(4.31) $$\left[V_{m+1}(\bar{n}+e_j) - V_m(\bar{n})\right]$$

$$=$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, p_{j0}\, [1-\alpha] +$$

$$h \sum_i \lambda_i \left[V_m(\bar{n}+e_i+e_j) - V_m(\bar{n}+e_i)\right] +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)} [1-\alpha] \sum_{\ell=0}^{N} p_{i\ell} \left[V_m(\bar{n}-e_i+e_\ell+e_j) - V_m(\bar{n}-e_i+e_\ell)\right] +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha] \sum_{\ell=1}^{N} p_{j\ell} \left[V_m(\bar{n}+e_\ell) - V_m(\bar{n})\right] +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [1-\alpha]\, p_{j0} \left[V_m(\bar{n}) - V_m(\bar{n})\right] +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)}\, [\alpha] \left[V_m(\bar{n}-e_i+e_j) - V_m(\bar{n}-e_i)\right] +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)]\, [\alpha] \left[V_m(\bar{n}) - V_m(\bar{n})\right] +$$

$$\left[1-h \sum_i \lambda_i - h \sum_i \mu_i(n_i) 1_{(n_i>0)} - h[\mu_j(n_j+1)-\mu_j(n_j)]\right] \left[V_m(\bar{n}+e_j)-V_m(\bar{n})\right].$$

Now first recall that $\mu_j(n_j)$ is nondecreasing for any j (see 2.1) and note that $h=Q^{-1}$ with Q given by (4.1), so that the coefficient in the last term is nonnegative, i.e.

$$\left[1 - h \sum_i \lambda_i - h \sum_i \mu_i(n_i) - h \, [\mu_j(n_j+1)-\mu_j(n_j)]\right] \geq 0 \, .$$

By substituting the lower estimates 0 from (4.28) for t=m in the right hand side of (4.31), we then immediately conclude: $V_{m+1}(\bar{n}+e_j)-V_{m+1}(\bar{n}) \geq 0$.
To estimate the right hand side of (4.31) from above, recall that the fifth term is equal to 0, while its coefficient is exactly equal to the first additional term:

$$h \left[\mu_j(n_j+1) - \mu_j(n_j)\right] p_{j0} \, [1-\alpha].$$

As a consequence, by substituting the upper estimates 1 from (4.28) for t=m and recalling again that all coefficients sum up to 1 by virtue of $h=Q^{-1}$ with Q defined by (4.1), we also obtain from (4.31): $V_{m+1}(\bar{n}+e_j)-V_{m+1}(\bar{n})\leq 1$.

The proof is thus completed by induction. □

LEMMA 2 Let

$$\Phi_1(\bar{n}) = 1_{(\bar{n}\in B)}$$

(4.32) $$\Phi_2(\bar{n}) = \sum_i \mu_i(n_i)$$

$$\Phi_3(\bar{n}) = 1_{(\bar{n}\in B)} \sum_i \mu_i(n_i)$$

and consider model 3 with α=0. Then for t≥0:

(4.33) $$T_t^2 \, \Phi_z(\bar{0}) \leq T_t^3 \, \Phi_z(\bar{0}) \qquad (z=1,2,3)$$

(4.34) $$T_t^3 \, \Phi_z(\bar{0}) \leq T_{t+1}^3 \, \Phi_z(\bar{0}) \leq \sum_{\bar{n}} \pi_0^3(\bar{n}) \, \Phi_z(\bar{n}) \qquad (z=1,2,3).$$

Proof of (4.33)

Let $M(S^3)$ be the set of all functions $f\colon S^3 \to \mathbb{R}$ such that

(4.35) $$f(\bar{n}+e_j) \geq f(\bar{n}) \qquad (\bar{n}\in S^3, \; j=1,\ldots,N).$$

We aim to prove that for any $f \in M(S^3)$:

(4.36) $$T_t^2 f(\bar{n}) \leq T_t^3 f(\bar{n}) \qquad (t \geq 0, \ \bar{n} \in S^2).$$

The proof of (4.33) would hereby be completed as $\Phi_z(.) \in M(S^3)$ for z=1,2,3.

To prove (4.36) we will apply induction on t. Clearly, (4.36) holds for t=0 as $T_0^2 f = T_0^3 f = f$. Suppose that (4.36) holds for $t \leq m$ and f satisfying (4.35). Then for t=m+1 and $\bar{n} \in S^2$ we can write:

(4.37) $$T_{m+1}^2 f(\bar{n}) - T_{m+1}^3 f(\bar{n}) =$$

$$T^2 (T_m^2 f)(\bar{n}) - T^3 (T_m^3 f)(\bar{n}) =$$

$$(T^2 - T^3) T_m^3 f(\bar{n}) + T^2 (T_m^2 f - T_m^3 f)(\bar{n}) ,$$

where the latter step is justified as before (see (4.19)) as $S^2 \subset S^3$. Also, as before (see (4.11), (4.20) and (4.25)), we note that T^2 is a positive operator (that is, $T^2 f \leq T^2 g$ if $f \leq g$ componentwise), while we recall (see (4.20)) that the transition matrix P^2 remains restricted to S^2. As a consequence, by induction hypothesis (4.36) for t=m and any $\bar{n} \in S^2$, from (4.37) we would also conclude for any $\bar{n} \in S^2$:

(4.38) $$T_{m+1}^2 f(\bar{n}) \leq T_{m+1}^3 f(\bar{n})$$

by proving

(4.39) $$(T^2 - T^3)(T_m^3 f)(\bar{n}) \leq 0 \qquad (\bar{n} \in S^2).$$

To prove this, for any function $g: S^3 \to \mathbb{R}$ we derive as in (4.22) for $\bar{n} \in S^2$:

(4.40) $$(T^3 - T^2)\, g(\bar{n}) =$$

$$\sum_i \lambda_i Q^{-1} 1_{(n_i = B_i)} [g(\bar{n} + e_i) - g(\bar{n})] +$$

$$\sum_i \mu_i(n_i) Q^{-1} \sum_j p_{ij} 1_{(n_j = B_j)} [g(\bar{n} - e_i + e_j) - g(\bar{n} - e_i)] \geq 0$$

provided $g \in M(S^3)$. Hence, inequality (4.39), and thus (4.38), would be proven provided:

$$T^3_m f \in M(S^3) \quad \text{for any } m \text{ and any } f \in M(S^3).$$

That is, for any t and $f \in M(S^3)$.

(4.41) $\quad T^3_t f(\bar{n}+e_j) - T^3_t f(\bar{n}) \geq 0 \qquad$ for all $\bar{n} \in S^2$, $j=1,\ldots,N$.

This in turn will again be proven by induction. Clearly, (4.41) holds for $m=0$ as $T^3_0 f = f$. Suppose that (4.41) holds for $t \leq m$. Then, similarly to (4.29)-(4.31), we derive for $t=m+1$ and with $h=Q^{-1}$:

(4.42)
$$\left[T^3_{m+1} f(\bar{n}+e_j) - T^3_{m+1} f(\bar{n})\right]$$

$$=$$

$$h \sum_i \lambda_i \left[T^3_m f(\bar{n}+e_i+e_j) - T^3_m f(\bar{n}+e_i)\right] +$$

$$h \sum_i \mu_i(n_i)\, 1_{(n_i>0)} \sum_{\ell=0}^N p_{i\ell} \left[T^3_m f(\bar{n}-e_i+e_\ell+e_j) - T^3_m f(\bar{n}-e_j+e_\ell)\right] +$$

$$h\, [\mu_j(n_j+1) - \mu_j(n_j)] \sum_{\ell=1}^N p_{j\ell} \left[T^3_m f(\bar{n}+e_\ell) - T^3_m f(\bar{n})\right] +$$

$$\left[1-h \sum_i \lambda_i - h \sum_i \mu_i(n_i) 1_{(n_i>0)} - h[\mu_j(n_j+1)-\mu_j(n_j)]\right]\left[T^3_m f(\bar{n}+e_j) - T^3_m f(\bar{n})\right]$$

Substituting the induction hypothesis: $T^3_m f \in M(S^3)$ and recalling $h=Q^{-1}$ with (4.1), the right hand side is estimated from below by 0, so that $T^3_{m+1} f \in M(S^3)$. By induction (4.41) is hereby proven. □

Proof of (4.34)

Let $M(S^3)$ as before be the set of all functions satisfying (4.35) at S^3. Recalling that $\Phi_z(.) \in M(S^3)$ for $z=1,2,3$, the first inequality of (4.34) will then be proven by showing that for any $f \in M(S^3)$:

(4.43) $\qquad T^3_t f(\bar{0}) \leq T^3_{t+1} f(\bar{0}) \qquad (t \geq 0).$

To apply induction on t, for t=0 and f satisfying (4.35) we obtain:

(4.44) $$T^3 f(\bar{0}) = h \sum_i \lambda_i f(\bar{0}+e_i) + [1-h \sum_i \lambda_i] f(\bar{0}) \geq f(\bar{0}).$$

Suppose that (4.43) holds for $t \leq m$ and all $f \in M(S^3)$. Then for $t=m+1$:

(4.45) $$T^3_{m+2} f(\bar{0}) - T^3_{m+1} f(\bar{0}) =$$

$$T^3_{m+1} (T^3 f)(\bar{0}) - T^3_m (T^3 f)(\bar{0}) \geq 0$$

by virtue of having proven (4.41) for t=1. By induction, (4.43) is thus verified for all t and $f \in M(S^3)$, which proves the first inequality of (4.34). The second inequality of (4.34) is an immediate consequence of monotone convergence for $t \to \infty$.

Remark (Unbouded intensities) We note that our proofs have assumed the transition rates to be uniformly bounded (see (4.1)). This may not apply in the infinite (model 1 and 3) cases when for example infinite server stations are involved. However, with more technical details, by applying an approximate uniformization method as in [14], the same results can be shown to remain valid also in such situations.

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