Appendix I

Analytical vibration model

For the model-updating procedure, an analytical vibration model was created in Matlab (Mathworks Inc., Natick, MA, USA) that contained five concentrated masses (the vertebrae) that were each connected by a hinge joint (the intervertebral disc and other soft tissue connections). The lower mass (L5) was clamped and the top mass (L1) was left free (Figure A.1). Each hinge had three rotational degrees of freedom which described flexion-extension, lateroflexion and torsion:

\[ \alpha_i = \begin{bmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{bmatrix} \] (A.1)

The rotation in the joints was defined by the rotation matrix \( R \). Assuming that rotations were small, sin and cos-terms could be neglected:

\[ R = R_y R_p R_a \equiv \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \] (A.2)

The coordinates of a joint relative to the previous joint and the coordinates of a centre of mass relative to the previous joint were respectively:

\[ \chi^j = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \chi^m = \begin{bmatrix} vL \\ 0 \\ 0 \end{bmatrix} \] (A.3), (A.4)

As a consequence the positions and accelerations of joints and centres of mass were defined as:

\[ \ddot{\chi}^j = \ddot{x}^j + R \dot{x}^j \rightarrow \ddot{x}^j = \ddot{x}^j + \ddot{R} \dot{x}^j \] (A.5)

\[ \ddot{\chi}^m = \ddot{x}^m + R \dot{x}^m \rightarrow \ddot{x}^m = \ddot{x}^m + \ddot{R} \dot{x}^m \] (A.6)
Figure A.1. Analytical model of a lumbar spine section. The “real” lumbar spine (left image) is potted at the lower end vertebra (L5) and L1 is left free, the model (right images A and B) is fixed at the lower end and free at the top. The model contains five concentrated masses which are each connected by a spring $k_1$-$k_4$, representing the intervertebral joints L1 to L5. The positions and accelerations $\mathbf{m}_x$, $\mathbf{m}_y$, $\mathbf{m}_z$ of each centre of mass are calculated relative to the positions and accelerations of the joint below each mass (panel B). The distance between each centre of mass and the joint below is vL and the distance between joints is L. The location of the centre of mass is given relative to the assumed rotation point in the centre of the vertebral body by $c_y$. 
Since angular velocities were small and higher order terms could be neglected, the accelerations of each vertebra’s centre of mass were obtained according to:

\[
\ddot{x}_m = \ddot{\mathbf{R}}_4 \ddot{x}_m = \begin{bmatrix} 0 & -\ddot{\gamma}_4 & \dddot{\beta}_4 & \nu L \\ \ddot{\gamma}_4 & 0 & -\dddot{\alpha}_4 & 0 \\ -\dddot{\beta}_4 & -\dddot{\alpha}_4 & 0 & 0 \end{bmatrix} \nu L \hat{\alpha}_4 \\
\ddot{x}_j = \ddot{\mathbf{R}}_4 \ddot{x}_j + \ddot{\mathbf{R}}_3 \ddot{x}_m
\]

\[
\begin{bmatrix} 0 & -\ddot{\gamma}_4 & \dddot{\beta}_4 & \nu L \\ \ddot{\gamma}_4 & 0 & -\dddot{\alpha}_4 & 0 \\ -\dddot{\beta}_4 & -\dddot{\alpha}_4 & 0 & 0 \end{bmatrix} \nu L = L \ddot{e} (\dddot{\alpha}_4 + \nu \dddot{\alpha}_4) \\
\ddot{x}_m = L \ddot{e} (\dddot{\alpha}_4 + \dddot{\alpha}_3 + \nu \dddot{\alpha}_4) \\
\ddot{x}_j = L \ddot{e} (\dddot{\alpha}_4 + \dddot{\alpha}_3 + \dddot{\alpha}_2 + \nu \dddot{\alpha}_4)
\]

With:

\[
\ddot{e} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}
\]

Equations A7-10 can be summarised by:

\[
\begin{bmatrix} \dddot{x}_m \\ \dddot{\alpha}_m \end{bmatrix} = T_i \dddot{\alpha}
\]

With:

\[
T_i = \begin{bmatrix} L \ddot{e} & 0 & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}, T_3 = \begin{bmatrix} L \ddot{e} & \nu L \ddot{e} & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix},
\]

\[
T_i = \begin{bmatrix} L \ddot{e} & L \ddot{e} & \nu L \ddot{e} & 0 \\ 0 & 0 & I & 0 \end{bmatrix}, T_3 = \begin{bmatrix} L \ddot{e} & L \ddot{e} & L \ddot{e} & \nu L \ddot{e} \\ 0 & 0 & 0 & I \end{bmatrix}
\]
Thus the virtual work by the inertia load of a vertebra was:

$$\delta W_i = \left[ \begin{array}{c} \delta \ddot{x}_i \\ \delta \ddot{\alpha}_i \\ \delta \ddot{\alpha}_i \end{array} \right] = \left[ \begin{array}{c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[ \begin{array}{c} F \\ M \end{array} \right] = \left[ \begin{array}{c} \delta \ddot{x}_i \\ \delta \ddot{\alpha}_i \end{array} \right] M \left[ \begin{array}{c} \ddot{x}_i \\ \ddot{\alpha}_i \end{array} \right]$$

(A.20)

$M$ can be expanded according to Equation A.21, with $m_i$ the mass of the vertebra, $J_i$ the mass moment of inertia, and $c_i$ the offset of the centre of mass from the geometrical centre of the vertebra:

$$M_i = \left[ \begin{array}{cc} m_i I & -m_i c_i \\ -m_i c_i^T & J_i + m_i c_i^2 \end{array} \right]$$

(A.21)

For virtual displacements holds (see Equation A.12):

$$\left[ \begin{array}{c} \delta \ddot{x}_i \\ \delta \ddot{\alpha}_i \end{array} \right] = T_i \alpha \rightarrow \left[ \begin{array}{c} \delta \ddot{x}_i \\ \delta \ddot{\alpha}_i \end{array} \right] = \delta \alpha^T T_i^T$$

(A.22)

Thus the virtual work by the inertia load of a vertebra was:

$$\delta W_i^V = \delta \alpha^T T_i^T M_i T_i \alpha \rightarrow \delta W_i^V = \delta \alpha^T M_i^p \ddot{\alpha}$$

(A.23)

Where $M_i^p$ is the projected mass matrix obtained by pre and post multiplying $M_i$ with $T$. The virtual work by the elastic moments in the joint was:

$$\delta W_e^V = \delta \alpha^T K \alpha$$

(A.24)

The equations of motion then become:

$$\left( M_i^p + M_j^p + M_k^p + M_l^p \right) \ddot{\alpha} + K \alpha = 0$$

(A.25)
In which $K$ is the stiffness-matrix that can be expanded according to:

$$
K = \begin{bmatrix}
  k_1 & -k_1 & 0 & 0 \\
  -k_1 & k_1 + k_2 & -k_2 & 0 \\
  0 & -k_2 & k_2 + k_3 & -k_3 \\
  0 & 0 & -k_3 & k_3 + k_4 \\
\end{bmatrix}
$$

(A.26)

The equations of motion for this model were solved in the modal space, which results in the eigenvalue problem:

$$
\left( -\omega^2 M + K \right) \Phi = 0
$$

(A.27)

where $\omega^2$ represents the eigenvalues from which the eigenfrequencies can be derived, and $\Phi$ the mode shape vector.