Appendix A

Technical Appendices

A.1 Uniform Integrability of Spacings

To derive the following lemmas, we require Assumption **(A1)**, **(A3)**, and **(A4)** in Chapter 2, or the analogous assumptions in Chapters 2 or 4. To remind, we require that our i.i.d. collection of continuous random variables $\mathbf{Z} = (Z_i, 1 \leq i \leq m)$ possess (at least) a finite second moment, and given the density function f_θ and quantile function $q_\alpha(\theta)$ of Z_1, we require that $\forall \theta \in \Theta \subset \mathbb{R}$ and $\alpha \in (0,1)$ that $f_\theta(q_\alpha(\theta)) > 0$.

In the Badahur method, [4], that is used in the almost sure convergence of an estimator to the value of the quantile, a key step is to ascertain a bound for the tail of the quantile estimator sequence. This is the bound provided for the order statistic in Lemma A.1. Lemma A.2 makes use of this identity and together with a criterion for uniform integrability, the uniform integrability of the spacing sequence $(Z_{\lceil \alpha m \rceil:m} - Z_{\lceil \alpha m \rceil - 1:m}, m \geq 1)$ is assured.

Lemma A.1 (Serfling, [94], pp.97)**.** *Let $\alpha \in (0,1)$ and suppose that Assumptions* **(A1)**, **(A3)** *and* **(A4)** *hold. Let l_n be a sequence of positive integers, with $1 \leq l_n \leq n$, such that*

$$\frac{l_n}{n} = \alpha + o\left((\ln(n))^\delta n^{-\frac{1}{2}}\right)$$

for some $\delta \geq 1/2$. Then with probability one

$$|Z_{l_n:n} - q_\alpha| \leq \frac{2}{f_\theta(q_\alpha)} (\ln(n))^\delta n^{-\frac{1}{2}}, \tag{A.1}$$

for all n sufficiently large.

We now turn to the proof of uniform integrability of spacings.

Lemma A.2 (Uniform Integrability of Spacings)**.** *Suppose that Assumptions* **(A1)**, **(A3)**, *and* **(A4)** *hold, then*

(i) *For $m \geq 1$, the normalized spacing sequences $(m(Z_{\lceil \alpha m \rceil:m} - Z_{\lceil \alpha m \rceil - 1:m}))$ and $(m^2(Z_{\lceil \alpha m \rceil:m} - Z_{\lceil \alpha m \rceil - 1:m})^2)$, $m \geq 1$, are uniformly integrable;*

(ii) *For* $a = 1, 2$ *and* $b > a$*, the sequence* $(m^a(Z_{\lceil \alpha m\rceil:m} - Z_{\lceil \alpha m\rceil -1:m})^b : m \geq 1)$ *tends to* 0 *a.s. as* $m \to \infty$.

Proof. To simplify the notation, let

$$Y_{1,m} = Z_{\lceil \alpha m\rceil -1:m}, \quad Y_{2,m} = Z_{\lceil \alpha m\rceil :m}.$$

Note that

$$Y_{2,m} - Y_{1,m} = F_\theta^{-1}(T_{2,m}) - F_\theta^{-1}(T_{1,m}),$$

where $T_{1,m} = U_{\lceil \alpha m\rceil -1:m}$, $T_{2,m} = U_{\lceil \alpha m\rceil :m}$ are taken from a uniform-(0,1) order statistics. Since F_θ is differentiable and monotone, there exist a $\xi \in (Y_{1,m}, Y_{2,m})$ such that by the Mean Value Theorem,

$$Y_{2,m} - Y_{1,m} = \frac{\partial}{\partial x} F_\theta^{-1}(\xi)(T_{2,m} - T_{1,m}) = \frac{1}{f_\theta(F_\theta^{-1}(\xi))}(T_{2,m} - T_{1,m}). \tag{A.2}$$

Let

$$r_m = \frac{2}{f_\theta(q_\alpha(\theta))}\left(\frac{\ln(m)}{m}\right)^{\frac{1}{2}}.$$

Note that letting $\delta = 1/2$ in Lemma A.1, implies

$$|q_\alpha(\theta) - Y_{2,m}| \leq r_m, \quad |q_\alpha(\theta) - Y_{1,m}| \leq r_m,$$

for all but finitely many m, which gives

$$|Y_{2,m} - Y_{1,m}| \leq |q_\alpha(\theta) - Y_{2,m}| + |q_\alpha(\theta) - Y_{1,m}| \leq 2r_m.$$

By **(A2)**, $f_\theta(x) > 0$ within a neighbourhood $B(\alpha)$ of q_α. Letting $B_{r_m}(q_\alpha(\theta))$ be the open ball centred at $x = q_\alpha(\theta)$ with radius r_m, it then holds for m sufficiently large that $B_{r_m}(q_\alpha) \subset B(\alpha)$, which implies

$$c_m = \inf_{y \in B_{r_m}(q_\alpha(\theta))} f_\theta(y), \quad C_m = \sup_{y \in B_{r_m}(q_\alpha(\theta))} f_\theta(y)$$

are finite. Inserting these constants into (A.2) we arrive at

$$\frac{1}{C_m^2}\mathbb{E}\left[(T_{2,m} - T_{1,m})^2\right] \leq \mathbb{E}_\theta\left[(Y_{2,m} - Y_{1,m})^2\right] \leq \frac{1}{c_m^2}\mathbb{E}\left[(T_{2,m} - T_{1,m})^2\right].$$

Note that by construction, it holds that

$$\lim_{m\to\infty} c_m = \lim_{m\to\infty} C_m = \frac{1}{f_\theta(q_\alpha(\theta))}. \tag{A.3}$$

Hence, taking the limit in (A.2) as m tends to infinity, it follows from the limit in (A.3) together with the fact that

$$\mathbb{E}\big[(T_{2,m} - T_{1,m})^2\big] = \frac{2}{m(m+1)}$$

for all m, that

$$\lim_{m\to\infty} \mathbb{E}_\theta\big[m^2(Y_{2,m} - Y_{1,m})^2\big] = \frac{2}{f_\theta^2(q_\alpha(\theta))}. \tag{A.4}$$

Noting that

$$\mathbb{E}[T_{2,m} - T_{1,m}] = \frac{1}{m+1},$$

it follows from the same line of arguments that

$$\lim_{m\to\infty} \mathbb{E}_\theta[m(Y_{2,m} - Y_{1,m})] = \frac{1}{f_\theta(q_\alpha(\theta))}. \tag{A.5}$$

We begin by firstly verifying uniform convergence, part (i). From **(A1)** together with [86] it follows that

$$m(Y_{2,m} - Y_{1,m}) \xrightarrow{d} \frac{E}{f_\theta(q_\alpha(\theta))}, \tag{A.6}$$

where E is an exponential random variable with mean one. Applying the mapping $g(x) = x^2$ to the term $m(Y_{2,m} - Y_{1,m})$, the Continuity Mapping Theorem yields

$$m^2(Y_{2,m} - Y_{1,m})^2 \xrightarrow{d} V := \frac{E^2}{f_\theta^2(q_\alpha(\theta))}, \tag{A.7}$$

with

$$\mathbb{E}[V] = \frac{2}{f_\theta^2(q_\alpha(\theta))}.$$

A sufficient condition for uniform integrability of a sequence $(X_n : n \geq 1)$ is that as $n \to \infty$, $X_n \xrightarrow{d} X$ and $\lim_n \mathbb{E}|X_n| = \mathbb{E}|X| < \infty$, see [94]. Hence, from (A.5) together with (A.6) follows the uniform integrability of $(m(Y_{2,m} - Y_{1,m}) : m \geq 1)$ and from (A.4) together with (A.6) and (A.7) follows uniform integrability of the sequence $(m^2(Y_{2,m} - Y_{1,m})^2)$.

We now turn to the proof of part (ii). The sequence, $(m^a(Y_{2,m}-Y_{1,m})^b : m \geq 1)$ is positive with probability one with mean

$$\mathbb{E}_\theta\left[m^a(Y_{2,m}-Y_{1,m})^b\right] \leq 2\left(\frac{\ln(m)}{m}\right)^{\frac{1}{2}(b-a)}\left(\frac{2}{f_\theta(q_\alpha(\theta))}\right)^{b-a} \cdot \mathbb{E}_\theta\left[m^a(Y_{2,m}-Y_{1,m})^a\right],$$

which follows from Lemma A.1, where we have assumed that $b > a$. Since for $a = 1, 2$, with

$$\mathbb{E}_\theta\left[m^a(Y_{2,m}-Y_{1,m})^a\right] = O(1),$$

this means

$$\mathbb{E}_\theta\left[m^a(Y_{2,m}-Y_{1,m})^b\right] \to 0$$

as $m \to \infty$. Combining these results with Property H in [96], p. 185, we arrive at

$$m^a(Y_{2,m}-Y_{1,m})^b \overset{a.s.}{\to} 0,$$

which concludes the proof for part (ii) of Lemma A.2. □

A.2 Uniform Order Statistic Expectations for extent of Asymptotic Unbiasedness Derivations

Below is the collection of results on expectations for functions of uniform-$(0,1)$ order statistics needed in Theorem 2.3 and Theorem 4.2. The random variables $T_1 := T_{[\lceil\alpha m\rceil-1]:m}$, $T_2 := T_{[\lceil\alpha m\rceil]:m}$ are the shorthand labels for the order statistics of an i.i.d. collection of $m \geq 1$ copies of uniform-$(0,1)$ random variables, $\alpha \in (0,1)$.

These computations were conducted via the Maple 14 or Maple 15 programming packages. To simplify, we make the approximation $\lceil\alpha m\rceil - \alpha m \approx 1$, since the difference of the coefficients are no larger than magnitude $O(1)$, and therefore not affecting the leading order behaviour.

By computation,

$$\mathbb{E}[T_2 - T_1] = \frac{1}{m+1}, \tag{A.8}$$

$$\mathbb{E}\left[(T_2 - T_1)^2\right] = \frac{2}{m(m+1)}. \tag{A.9}$$

$$\begin{aligned}\mathbb{E}[(T_2 - T_1)(T_1 - \alpha)] &= \frac{(\lceil\alpha m\rceil - \alpha m) - (\alpha+1)}{m(m+1)} \\ &\approx -\frac{\alpha}{m(m+1)},\end{aligned} \tag{A.10}$$

$$
\begin{aligned}
\mathbb{E}[(T_2 - T_1)(T_2 - \alpha)] &= \frac{(\lceil \alpha m \rceil - \alpha m) + 1 - \alpha}{m(m+1)}, \\
&\approx \frac{2-\alpha}{m(m+1)}. \qquad \text{(A.11)} \\
\mathbb{E}[(T_2 - T_1)(T_2 + T_1 - 2\alpha)] &= \frac{2(\lceil \alpha m \rceil - \alpha m) - 2\alpha)}{m(m+1)} \\
&\approx \frac{2(1-\alpha)}{m(m+1)}, \qquad \text{(A.12)}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\big[(T_2 - T_1)^2(T_2 + T_1 - 2\alpha)\big] \\
&\quad = \frac{4(\lceil \alpha m \rceil - \alpha m) - 8\alpha + 2}{m(m+1)(m+2)} \\
&\quad \approx \frac{6 - 8\alpha}{m(m+1)(m+2)}. \qquad \text{(A.13)} \\
&\mathbb{E}[(T_2 - T_1)(T_1 - \alpha)(T_2 + T_1 - 2\alpha)] \\
&\quad = \frac{2\alpha(1-\alpha)m + 2(\lceil \alpha m \rceil - \alpha m)^2 + 8\alpha(\lceil \alpha m \rceil - \alpha m) + 4\alpha^2 + 4\alpha - 2}{m(m+1)(m+2)} \\
&\quad \approx \frac{2\alpha(1-\alpha)}{(m+1)(m+2)} + \frac{4\alpha(3+\alpha)}{m(m+1)(m+2)}, \qquad \text{(A.14)} \\
&\mathbb{E}[(T_2 - T_1)(T_2 - \alpha)(T_2 + T_1 - 2\alpha)] \\
&\quad = \frac{2\alpha(1-\alpha)m + 2(\lceil \alpha m \rceil - \alpha m)^2 + 4(1-2\alpha)(\lceil \alpha m \rceil - \alpha m) - 4\alpha(1-\alpha)}{m(m+1)(m+2)} \\
&\quad \approx \frac{2\alpha(1-\alpha)}{(m+1)(m+2)} + \frac{2(1-2\alpha)(3-2\alpha)}{m(m+1)(m+2)} \qquad \text{(A.15)} \\
&\mathbb{E}\big[(T_2 - T_1)^2(T_2 + T_1 - 2\alpha)^2\big], \\
&\quad = \frac{8\alpha(1-\alpha)m + 8(\lceil \alpha m \rceil - \alpha m)^2 + 16(1-3\alpha)(\lceil \alpha m \rceil - \alpha m) - 24\alpha(1-2\alpha)}{m(m+1)(m+2)(m+3)} \\
&\quad \approx \frac{8\alpha(1-\alpha)}{(m+1)(m+2)(m+3)} + \frac{24(1-\alpha)(1-2\alpha)}{m(m+1)(m+2)(m+3)}. \qquad \text{(A.16)}
\end{aligned}
$$

A.3 Verification of Assumptions of Call Option Quantile Sensitivities

Let F_θ be the distribution function and f_θ the density function, $\theta \in \Theta \subset \mathbb{R}$ of the call option, and let χ be the density function of the underlying financial models. The function $\chi = \phi, \phi^{\mathrm{VG}}$, represents the densities of either the Black-Scholes-Merton (BSM) or Variance Gamma (VG) processes in Section 4.6.1. In addition, let $Z = h(X)$ represent the performance function of the call option. Verification of the call option distribution from either of the financial models stems from the expression

$$F_\theta(z) = \int_{\mathbb{R}} 1\{h(x) \le z\}\, \chi(x)\, dx.$$

The support for both processes is the entire real line. From the above equation, attaining derivatives from the distribution of the call option, or any performance function, is then simply the equivalent of differentiating the underlying process.

This appendix directly verifies Assumption **(C2)** for the quantile sensitivity Greeks Delta and Vega in Section 4.6.2.1 for both financial models. Though not needed, this appendix also directly verifies Assumption **(C4)**. These financial Greeks otherwise refer to the parameter sensitivities w.r.t. the current price s_0, and implied volatility σ. The verification of the α-quantile Delta, $\alpha \in (0,1)$ is provided in Section A.3.1 before the verification of the α-quantile Vega in Section A.3.2.

From Section 4.6.1, for $t > 0$, the BSM model $X(t)$ has a marginal normal distribution $X(t) = N(a(t), b(t))$ with mean $a(t)$ and standard deviation $b(t)$. The form of these parameters are given by

$$a(t) = \ln s_0 + \left(r - \frac{1}{2}\sigma^2\right) t \quad \text{and} \quad b(t) = \sigma\sqrt{t}.$$

Let $\tau(t) = \gamma(t/\nu, 1/\nu)$ signify a gamma distribution with scale parameter t/ν, and shape parameter $1/\nu$, $\nu > 0$. Instead, the marginal distribution of the Variance-Gamma process is a gamma distributed normal mean-variance mixture. Contingent on $\tau(t) = y$, let

$$\alpha(y) := \alpha_t(y) = \ln s_0 + (r+\omega)t + \kappa y, \quad \text{and} \quad \beta(y) := \beta_t(y) = \sigma\sqrt{y}$$

be the parameters for this normal basis. Then the marginal distribution, for fixed $t > 0$, for the Variance Gamma process is $Y(\tau(t)) = N(\alpha(\tau(t)), \beta(\tau(t)))$.

A.3.1 α-Quantile Delta

For both Greeks, we begin by showing that Assumptions **(C2)** and **(C4)** are satisfied for the BSM process before proceeding to the VG process. For the normal density function

$$\frac{\partial}{\partial x}\phi_{a(t),b(t)}(x) = -\frac{1}{\sqrt{2\pi}}\frac{x - a(t)}{b^3(t)} \exp\left(-\frac{1}{2}\left(\frac{x - a(t)}{b(t)}\right)^2\right),$$

which yields

$$\sup_{x\in\mathbb{R}} \left|\frac{\partial}{\partial x}\phi_{a(t),b(t)}(x)\right| = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2}}.$$

Also,

$$\frac{\partial}{\partial s_0}\phi_{a(t),b(t)}(x) = \frac{1}{\sqrt{2\pi}s_0}\frac{x-a(t)}{b^3(t)}\exp\left(-\frac{1}{2}\left(\frac{x-a(t)}{b(t)}\right)^2\right),$$

which attains

$$\sup_{x\in\mathbb{R}}\left|\frac{\partial}{\partial s_0}\phi_{a(t),b(t)}(x)\right| = \frac{1}{\sqrt{2\pi}\sigma^2 t s_0}e^{-\frac{1}{2}},$$

verifying the assumptions for the BSM model.

For the VG-process, we need to show that the supremum of the derivative of the density over a neighbourhood $B(\alpha)$ of the quantile is finite for a sufficiently large value of m. For $t>0$ it holds that

$$\begin{aligned}\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t) &= \int_0^\infty \frac{\partial}{\partial x}\phi_{\alpha(y),\beta(y)}(x)\,\frac{y^{\frac{t}{\nu}-1}e^{-\frac{y}{\nu}}}{\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\,dy\\ &= -\int_0^\infty \frac{1}{\sqrt{2\pi}}\frac{x-\alpha(y)}{\beta^3(y)}\exp\left(-\frac{1}{2}\left(\frac{x-\alpha(y)}{\beta(y)}\right)^2\right)\frac{y^{\frac{t}{\nu}-1}e^{-\frac{y}{\nu}}}{\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\,dy,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial s_0}\phi^{\mathrm{VG}}(x,t) &= \int_0^\infty \frac{1}{\sqrt{2\pi}s_0}\frac{x-\alpha(y)}{\beta^3(y)}\exp\left(-\frac{1}{2}\left(\frac{x-\alpha(y)}{\beta(y)}\right)^2\right)\frac{y^{\frac{t}{\nu}-1}e^{-\frac{y}{\nu}}}{\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\,dy\\ &= -\frac{1}{s_0}\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t).\end{aligned}$$

By determining the supremum by a naïve approach, applying Fubini's theorem and evoking the global supremum of the normal distribution will lead to the expectation $\mathbb{E}[\tau^{-1/2}]$ for the Gamma distribution $\gamma(t/\nu,1/\nu)$. This only leads to a finite value if $\nu > t/2$ for fixed t.

For this derivation, since the two derivatives are related, we will only look at the case for the derivative w.r.t. x. Recall that $\eta(x) = x-\ln(s_0)-(\mu+\omega)t$, Equation (4.27) in Section 4.6.2.1, such that for a fixed value of $\tau(t) = y$,

$$x-\alpha(y) = \eta(x)-\kappa y, \tag{A.17}$$

from which it follows that $x-\alpha(y) > 0$ is equivalent to $y < \eta(x)/\kappa$. The region in which we determine the local supremum of $\phi^{\mathrm{VG}}(x,t)$ is $B_{r_m}(q_\alpha) \subset B(\alpha)$ as defined in Lemma A.2. For m sufficiently large, it holds that

$$q_\alpha - r_m \le x \le q_\alpha + r_m.$$

Subtracting $\ln(s_0) + (\mu + \omega)t$ from the above row of inequalities leads to

$$\eta(q_\alpha) - r_m \leq \eta(x) \leq \eta(q_\alpha) + r_m. \tag{A.18}$$

Before turning to the actual derivation, we define, for notational convenience,

$$\zeta_\xi(x) = \exp\left(\frac{\kappa}{\sigma^2}\eta(x)\right)\left(\frac{\eta^2(x)}{\kappa^2 + \frac{2\sigma^2}{\nu}}\right)^{\frac{\xi}{2}} K_\xi\left(\frac{1}{\sigma^2}\sqrt{\eta^2(x)\left(\kappa^2 + \frac{2\sigma^2}{\nu}\right)}\right),$$

and for the derivation, given parameters $\nu, \beta > 0, \gamma > 0$, we require the integral result 3.471.9 in [44], given below:

$$\int_0^\infty x^{\nu-1} \exp\left(-\frac{\beta}{x} - \gamma x\right) dx = 2\left(\frac{\beta}{\gamma}\right)^{\frac{\nu}{2}} K_\nu\left(2\sqrt{\beta\gamma}\right). \tag{A.19}$$

Now we can ascertain the suprema. Writing g_ν as the Lebesgue density for the gamma function, noting $x - \alpha(y) = \eta(x) - \kappa y$, Equation (A.17), it follows that

$$\begin{aligned}\left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t)\right| &\leq \int_0^\infty \frac{1}{\sqrt{2\pi}} \frac{|x-\alpha(y)|}{\beta^3(y)} \exp\left(-\frac{1}{2}\left(\frac{x-\alpha(y)}{\beta(y)}\right)^2\right) g_\nu(y)\,dy \\ &\leq \frac{1}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}} \int_0^\infty |\eta(x) - \kappa y| \exp\left(-\frac{1}{2}\frac{(\eta(x)-\kappa y)^2}{\sigma^2 y}\right) y^{\frac{t}{\nu}-\frac{5}{2}} e^{-\frac{y}{\nu}}\,dy.\end{aligned} \tag{A.20}$$

A crucial step for computing the expression in (A.20) is the computation of bounds for

$$|\eta(x) - \kappa y| \quad \text{and} \quad \exp\left(-\frac{1}{2}\frac{(\eta(x)-\kappa y)^2}{\sigma^2 y}\right),$$

where we distinguish between the cases when both $\eta(q_\alpha)$ and κ are positive or negative; when one of $\eta(q_\alpha)$ and κ is positive and the other, negative, and $\kappa = 0$. We will also require for m to be sufficiently large such that if

$$\eta(q_\alpha) - \kappa y > 0 \quad \text{this implies} \quad \eta(q_\alpha) - r_m - \kappa y > 0,$$

and if

$$\eta(q_\alpha) - \kappa y < 0 \quad \text{this implies} \quad \eta(q_\alpha) + r_m - \kappa y < 0.$$

First, we consider the case $\eta(q_\alpha) > 0$ and $\kappa > 0$. When both parameters are negative, the upper bound is similar. Within the region $B_{r_m}(q_\alpha)$ it holds that

$$\eta(x) + \kappa y \leq \eta(q_\alpha) + r_m + \kappa y,$$

see (A.18). To maximize the exponential term, we need to split up the integral into the regions $\eta(q_\alpha)-\kappa y<0$ and $\eta(q_\alpha)-\kappa y>0$. In the first interval, $\eta(x)-\kappa y\leq\eta(q_\alpha)+r_m-\kappa y$, and, for m sufficiently large, $\eta(q_\alpha)-\kappa y<0$ implies $\eta(q_\alpha)+r_m-\kappa y<0$, from which it follows $(\eta(x)-\kappa y)^2>(\eta(q_\alpha)+r_m-\kappa y)^2$. For the case $\eta(q_\alpha)-\kappa y>0$ we attain by similar arguments $(\eta(x)-\kappa y)^2>(\eta(q_\alpha)-r_m-\kappa y)^2$. These considerations yield a supremum

$$\begin{aligned}
&\sup_{x\in B_{r_m}(q_\alpha)}\left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t)\right|\\
&\leq\frac{1}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\left(\int_0^{\frac{\eta(q_\alpha)}{\kappa}}y^{\frac{t}{\nu}-\frac{5}{2}}(\eta(q_\alpha)+r_m+\kappa y)\exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)-r_m-\kappa y)^2}{\sigma^2 y}\right)e^{-\frac{y}{\nu}}\,dy\right.\\
&\quad\left.+\int_{\frac{\eta(q_\alpha)}{\kappa}}^{\infty}y^{\frac{t}{\nu}-\frac{5}{2}}(\eta(q_\alpha)+r_m+\kappa y)\exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)+r_m-\kappa y)^2}{\sigma^2 y}\right)e^{-\frac{y}{\nu}}\,dy\right)\\
&\leq\frac{1}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\Bigg(\eta(q_\alpha+r_m)\Bigg(\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha-r_m)\right)\\
&\quad\cdot\int_0^{\infty}y^{\frac{t}{\nu}-\frac{5}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha-r_m))}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)\,dy\\
&\quad+\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha+r_m)\right)\int_0^{\infty}y^{\frac{t}{\nu}-\frac{5}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha+r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)\,dy\Bigg)\\
&\quad+\kappa\Bigg(\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha-r_m)\right)\int_0^{\infty}y^{\frac{t}{\nu}-\frac{3}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha-r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)\,dy\\
&\quad+\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha+r_m)\right)\int_0^{\infty}y^{\frac{t}{\nu}-\frac{3}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha+r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)\,dy\Bigg)\Bigg)\\
&=\frac{2}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\Big(\eta(q_\alpha+r_m)\Big(\zeta_{\frac{t}{\nu}-\frac{3}{2}}(q_\alpha-r_m)+\zeta_{\frac{t}{\nu}-\frac{3}{2}}(q_\alpha+r_m)\Big)\\
&\quad+\kappa\Big(\zeta_{\frac{t}{\nu}-\frac{1}{2}}(q_\alpha-r_m)+\zeta_{\frac{t}{\nu}-\frac{1}{2}}(q_\alpha+r_m)\Big)\Big).
\end{aligned}$$

Next, we discuss the case $\eta(q_\alpha)<0$ and $\kappa>0$, i.e., the case that $\eta(q_\alpha)-\kappa y<0$ for all $y>0$. For m sufficiently large, $\eta(q_\alpha)-\kappa y<0$ implies that $\eta(x)-\kappa y<0$ for all $y>0$, within $B_{r_m}(q_\alpha)$. As a result, $\eta(x)-\kappa y<\kappa y-\eta(x)<\kappa y+|\eta(q_\alpha)|+r_m$. Additionally, for the exponent, $(\eta(x)-\kappa y)^2>(\eta(q_\alpha)+r_m-\kappa y)^2$. Putting this altogether, we have the expression

$$\begin{aligned}
&\sup_{x\in B_{r_m}(q_\alpha)}\left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t)\right|\\
&\leq\frac{1}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\int_0^{\infty}y^{\frac{t}{\nu}-\frac{5}{2}}(\kappa y+|\eta(q_\alpha)|+r_m)\exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)+r_m-\kappa y)^2}{\sigma^2 y}\right)e^{-\frac{y}{\nu}}\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha + r_m)\right)}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}} \Big((|\eta(q_\alpha)| + r_m) \\
&\cdot \int_0^\infty y^{\frac{t}{\nu}-\frac{5}{2}} \exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha + r_m)}{\sigma^2 y}\right) \exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2} + \frac{1}{\nu}\right)y\right) dy \\
&+ \kappa \int_0^\infty y^{\frac{t}{\nu}-\frac{3}{2}} \exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha + r_m)}{\sigma^2 y}\right) \exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2} + \frac{1}{\nu}\right)y\right) dy\Big),
\end{aligned}
$$

in which we then arrive at the supremum

$$
\begin{aligned}
&\sup_{x \in B_{r_m}(q_\alpha)} \left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}_{\alpha(t),\beta(t)}(x,t)\right| \\
&\quad = \frac{2}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}} \Big((|\eta(q_\alpha)| + r_m)\,\zeta_{\frac{t}{\nu}-\frac{3}{2}}(q_\alpha + r_m) + \kappa\,\zeta_{\frac{t}{\nu}-\frac{1}{2}}(q_\alpha + r_m)\Big).
\end{aligned}
$$

For the case $\kappa = 0$, we attain a simpler expression. We note that within the open ball $B_{r_m}(q_\alpha)$, accounting for the sign of $\eta(q_\alpha)$, $\eta(x) \le |\eta(q_\alpha)| + r_m$ and for the exponential term, $\eta^2(x) \ge (|\eta(q_\alpha)| - r_m)^2$. We again have made the assumption that if $\eta(q_\alpha) > 0$, m is chosen sufficiently large such that $\eta(q_\alpha) - r_m > 0$. A similar assumption holds if $\eta(q_\alpha) < 0$. The suprema now has the form

$$
\sup_{x \in B_{r_m}(q_\alpha)} \left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t)\right| \le \frac{|\eta(q_\alpha)| + r_m}{\sqrt{2\pi}\sigma^3\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}} \int_0^\infty y^{\frac{t}{\nu}-\frac{5}{2}} \exp\left(-\frac{1}{2}\frac{(|\eta(q_\alpha)| - r_m)^2}{\sigma^2 y}\right) e^{-\frac{y}{\nu}}\, dy,
$$

and after some rearrangement of the constants and the use of the same integral expression

$$
\sup_{x \in B_{r_m}(q_\alpha)} \left|\frac{\partial}{\partial x}\phi^{\mathrm{VG}}(x,t)\right| = \frac{2^{\frac{5}{4}}(|\eta(q_\alpha)| + r_m)^{\frac{t}{\nu}-\frac{1}{2}}}{\sqrt{\pi}\sigma^{\frac{3}{2}}\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{3}{4}}(2\nu\sigma^2)^{\frac{t}{2\nu}}} K_{\frac{t}{\nu}-\frac{3}{2}}\left(\sqrt{\frac{2}{\nu}}\frac{|\eta(q_\alpha)| - r_m}{\sigma}\right).
$$

Note that while $\eta(q_\alpha) = 0$ is possible this event has probability zero. The case $\kappa = 0$ corresponds to the symmetric Variance Gamma process.

A.3.2 α-Quantile Vega

The method to attain the partial derivative $\partial_\sigma\phi(x)$ for a BSM process is as described in Section A.3.1. The absolute supremum for this density function is evident.

Moreover, The equivalent method for the α-quantile Delta for the VG process is again implemented. Ensuring that each term in the normal derivative is positive, we have the expression

$$\left|\frac{\partial}{\partial\sigma}\phi^{\mathrm{VG}}(x,t)\right| = \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma\beta(y)}\left(\frac{(x-\alpha(y))^2}{\beta^2(y)}+1\right)\exp\left(-\frac{1}{2}\left(\frac{x-\alpha(y)}{\beta(y)}\right)^2\right) g_\nu(y)\,dy$$
$$+\int_0^\infty \frac{\sigma t e^{-\nu\omega}}{\sqrt{2\pi}\beta(y)}\frac{|\eta(x)-\kappa y|}{\beta^2(y)}\exp\left(-\frac{1}{2}\left(\frac{x-\alpha(y)}{\beta(y)}\right)^2\right) g_\nu(y)\,dy,$$

in which $e^{-\nu\omega} = (1-\kappa\nu-\sigma^2\nu/2)^{-1}$. Though the technique to determine a bound for the the supremum is the same as in the previous example, the process is more involved. We will only consider here the more difficult case $\eta(q_\alpha)/\kappa > 0$ where both quantities are positive.

In this case, the supremum for the partial derivative w.r.t σ has the form

$$\sup_{x\in B_{r_m}(q_\alpha)}\left|\frac{\partial}{\partial\sigma}\phi^{\mathrm{VG}}(x,t)\right|$$
$$= \frac{2}{\sqrt{2\pi}\sigma^2\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\left((\eta(q_\alpha)+r_m)\left(\frac{\eta(q_\alpha+r_m)}{\sigma^2}+te^{-\nu\omega}\right)\left(\zeta_{\frac{t}{\nu}-\frac{3}{2}}(q_\alpha+r_m)\right.\right.$$
$$\left.+\zeta_{\frac{t}{\nu}-\frac{3}{2}}(q_\alpha-r_m)\right) + \left(\frac{2\kappa\eta(q_\alpha+r_m)}{\sigma^2}+\kappa te^{-\nu\omega}+1\right)\zeta_{\frac{t}{\nu}-\frac{1}{2}}(q_\alpha+r_m)$$
$$\left.+\left(\kappa te^{-\nu\omega}+1\right)\zeta_{\frac{t}{\nu}-\frac{1}{2}}(q_\alpha-r_m)+\frac{\kappa^2}{\sigma^2}\left(\zeta_{\frac{t}{\nu}+\frac{1}{2}}(q_\alpha+r_m)+\zeta_{\frac{t}{\nu}+\frac{1}{2}}(q_\alpha-r_m)\right)\right).$$

In the following we provide the details on how to attain the above bound. For the derivation, within the integrand, we observe that $|\eta(x)-\kappa y| \le \eta(q_\alpha)+r_m+\kappa y$ in the interval $B_{r_m}(q_\alpha)$. Also, we need to split both integrals into two intervals: $\eta(q_\alpha)-\kappa y > 0$ and $\eta(q_\alpha)-\kappa y < 0$. For the first interval, in $B_{r_m}(q_\alpha)$,

$$(\eta(q_\alpha)-r_m-\kappa y)^2 \le (\eta(x)-\kappa y)^2 \le (\eta(q_\alpha)+r_m-\kappa y)^2.$$

The lower bound is for the exponent and the upper bound is for the quadratic term. For $\eta(q_\alpha)-\kappa y < 0$

$$(\eta(q_\alpha)+r_m-\kappa y)^2 \le (\eta(x)-\kappa y)^2 \le (\eta(q_\alpha)-r_m-\kappa y)^2.$$

Inserting these bounds into the expression $|\partial_\sigma\phi^{\mathrm{VG}}(x,t)|$, we get our first bound of the supremum

$$\sup_{x\in B_{r_m}(q_\alpha)}\left|\frac{\partial}{\partial\sigma}\phi^{\mathrm{VG}}(x,t)\right|$$
$$\le \int_0^{\frac{\eta(q_\alpha)}{\kappa}} \frac{1}{\sqrt{2\pi}\sigma^2\sqrt{y}}\exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)-r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy$$
$$+\int_{\frac{\eta(q_\alpha)}{\kappa}}^\infty \frac{1}{\sqrt{2\pi}\sigma^2\sqrt{y}}\exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)+r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy$$

$$+\int_0^{\frac{\eta(q_\alpha)}{\kappa}} \frac{1}{\sqrt{2\pi}\sigma^4\sqrt{y^3}}(\eta(q_\alpha)+r_m-\kappa y)^2 \exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)-r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy$$
$$+\int_{\frac{\eta(q_\alpha)}{\kappa}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma^4\sqrt{y^3}}(\eta(q_\alpha)-r_m-\kappa y)^2 \exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)+r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy$$
$$+\int_0^{\frac{\eta(q_\alpha)}{\kappa}} \frac{te^{-\nu\omega}}{\sqrt{2\pi}\sigma^2\sqrt{y^3}}(\eta(q_\alpha)+r_m+\kappa y) \exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)-r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy$$
$$+\int_{\frac{\eta(q_\alpha)}{\kappa}}^{\infty} \frac{te^{-\nu\omega}}{\sqrt{2\pi}\sigma^2\sqrt{y^3}}(\eta(q_\alpha)+r_m+\kappa y) \exp\left(-\frac{1}{2}\frac{(\eta(q_\alpha)+r_m-\kappa y)^2}{\sigma^2 y}\right) g_\nu(y)\,dy.$$

For the third integral, knowing $\eta(q_\alpha)-\kappa y>0$,

$$(\eta(q_\alpha)+r_m-\kappa y)^2 \le \eta^2(q_\alpha+r_m)+\kappa^2 y^2$$

since $\eta(q_\alpha)+r_m>0$, where we use the fact that $\eta(x+y)=\eta(x)+y$, Equation (4.27). For the fourth integral, where $\eta(q_\alpha)-\kappa y<0$, $\eta(q_\alpha)-r_m$ may be a negative value and so

$$(\eta(q_\alpha)-r_m-\kappa y)^2 < (\eta(q_\alpha)+r_m+\kappa y)^2 \;=\; \eta^2(q_\alpha+r_m)+2\kappa\eta(q_\alpha+r_m)y+\kappa^2 y^2,$$

separating the components. Extending the intervals for each of the integrals to $[0,\infty)$, and rearranging the integrals, we bound above again by

$$\le \frac{1}{\sqrt{2\pi}\sigma^2\Gamma\left(\frac{t}{\nu}\right)\nu^{\frac{t}{\nu}}}\left\{\eta(q_\alpha+r_m)\left(\frac{\eta(q_\alpha+r_m)}{\sigma^2}+te^{-\nu\omega}\right)\right.$$
$$\cdot\left(\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha+r_m)\right)\int_0^\infty y^{\frac{t}{\nu}-\frac{5}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha+r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy\right.$$
$$\left.+\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha-r_m)\right)\int_0^\infty y^{\frac{t}{\nu}-\frac{5}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha-r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy\right)$$
$$+\left(\frac{2\kappa\eta(q_\alpha+r_m)}{\sigma^2}+\kappa te^{-\nu\omega}+1\right)\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha+r_m)\right)$$
$$\cdot\int_0^\infty y^{\frac{t}{\nu}-\frac{3}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha+r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy$$
$$+\left(\kappa te^{-\nu\omega}+1\right)\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha-r_m)\right)\int_0^\infty y^{\frac{t}{\nu}-\frac{3}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha-r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy$$
$$+\frac{\kappa^2}{\sigma^2}\left(\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha+r_m)\right)\int_0^\infty y^{\frac{t}{\nu}-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha+r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy\right.$$
$$\left.\left.+\exp\left(\frac{\kappa}{\sigma^2}\eta(q_\alpha-r_m)\right)\int_0^\infty y^{\frac{t}{\nu}-\frac{1}{2}}\exp\left(-\frac{1}{2}\frac{\eta^2(q_\alpha-r_m)}{\sigma^2 y}\right)\exp\left(-\left(\frac{1}{2}\frac{\kappa^2}{\sigma^2}+\frac{1}{\nu}\right)y\right)dy\right)\right\}.$$

From Equation (A.19) and the definition of $\zeta_{(\cdot)}$, the final result is attained.

Appendix B

Measure-Valued Derivatives for the Bivariate Normal Distribution

This appendix determines the measure-valued derivatives w.r.t. the parameters present in the bivariate normal distribution, completing the results put forward in Section 4.6.2.2 where a partial derivative w.r.t. the common correlation ρ between two securities is discussed.

In Section B.1, we begin by introducing the notation and method, and reminding of the result of the parameter derivative w.r.t. ρ between two random variables X_1, X_2. Then via the chain rule, presented in Section 1.2.4.2, we also ascertain the parameter derivative w.r.t. mean and standard deviation of a marginal random variable. For all the three sensitivities, the generation of the measure-valued derivative random variables following the precepts within Section 1.2.4.2 are also shown. In Section B.2, we provide an alternative derivation to obtain a measure-valued derivative for the bivariate normal distribution that obtains the parameter derivative w.r.t. ρ and the marginal standard deviation.

B.1 Measure-Valued Derivatives

Let $X = (X_1, X_2)^{\mathsf{T}} \in \mathbb{R}^2$, where $(\cdot)^{\mathsf{T}}$ denotes the transpose of a vector, be a vector where each element is a normally distributed random variable, i.e. $X_l = N(a_l, b_l)$, $l = 1,2$, with mean $a_l \in \mathbb{R}$ and standard deviation $b_l > 0$. For the bivariate normal distribution, let $X \sim N(a, \Sigma)$ with $a = (a_1, a_2)^{\mathsf{T}}$ as the mean vector and Σ the 2×2 covariance matrix. Since Σ is symmetric, hence positive definite, the inverse Σ^{-1} exists, and the entries for both Σ and Σ^{-1} are given by

$$\Sigma = \begin{pmatrix} b_1^2 & \rho b_1 b_2 \\ \rho b_1 b_2 & b_2^2 \end{pmatrix} \qquad \Sigma^{-1} = \frac{1}{1-\rho^2} \begin{pmatrix} \frac{1}{b_1^2} & -\frac{\rho}{b_1 b_2} \\ -\frac{\rho}{b_1 b_2} & \frac{1}{b_2^2} \end{pmatrix}.$$

The determinant of Σ is given by $\det \Sigma = b_1^2 b_2^2 (1-\rho^2)$. Our results assume that the correlation[1] between these two random variables satisfies $\rho \in (-1, 1)$. The

[1] For either of the singular cases $\rho = 1, -1$, the result can either be attained from the appropriate limit, or, more simply, reformulate the problem in the sub-space $X_2 = pX_1 + q$ for constants p, q.

density function for the bivariate normal distribution $\phi_{a,\Sigma}(x)$, $x = (x_1, x_2)^\top$ is written as

$$\begin{aligned}\phi_{a,\Sigma}(x) &= \frac{1}{2\pi\sqrt{\det\Sigma}}\exp\left(-\frac{1}{2}(x-a)^\top\Sigma^{-1}(x-a)\right)\\ &= \frac{1}{2\pi b_1 b_2\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\frac{\left(\frac{x_1-a_1}{b_1}\right)^2 - 2\rho\left(\frac{x_1-a_1}{b_1}\right)\left(\frac{x_2-a_2}{b_2}\right)+\left(\frac{x_2-a_2}{b_2}\right)^2}{1-\rho^2}\right).\end{aligned}$$

The respective marginal mean and standard deviation that we are analysing is a_1 and b_1, that is, we are ascertaining a measure-valued derivative expression for each of the parameters ρ, a_1, and b_1. These results are attained, shown below, by rewriting the above density into the conditional form by attaining a realization for X_2 before acquiring a realization X_1 given X_2:

$$\begin{aligned}\phi_{a,\Sigma}(x) &= \phi_{\mu,\sigma}(x_1|x_2)\,\phi_{a_2,b_2}(x_2)\\ &= \frac{1}{2\sqrt{\pi}b_1\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\frac{\left(\left(\frac{x_1-a_1}{b_1}\right)-\rho\left(\frac{x_2-a_2}{b_2}\right)\right)^2}{1-\rho^2}\right)\\ &\quad\cdot\frac{1}{\sqrt{2\pi}b_2}\exp\left(-\frac{1}{2}\left(\frac{x_2-a_2}{b_2}\right)^2\right)\\ &= \frac{1}{2\sqrt{\pi}b_1\sqrt{1-\rho^2}}\exp\left(-\frac{1}{2}\left(\frac{x_1-(a_1+\rho b_1\left(\frac{x_2-a_2}{b_2}\right)}{b_1\sqrt{1-\rho^2}}\right)^2\right)\\ &\quad\cdot\frac{1}{\sqrt{2\pi}b_2}\exp\left(-\frac{1}{2}\left(\frac{x_2-a_2}{b_2}\right)^2\right).\end{aligned}\tag{B.1}$$

In Equation (B.1) the parameters ρ, a_1, and b_1 are solely present in the conditional density function $\phi_{\mu,\sigma}(x_1|x_2)$ and discerning the parameter derivative is reduced to differentiating this function. The terms $\mu := \mu(x_2)$ and σ define the mean and standard deviation for the conditional density and encapsulate the other parameters via

$$\mu(x_2) = a_1 + \rho b_1\frac{x_2-a_2}{b_2}, \quad \text{and} \quad \sigma = b_1\sqrt{1-\rho^2}.$$

In generation, we note that μ is a random variable as it is a function of the random variable X_2.

As discussed in Section 4.6.2.2, the correlation ρ is present in both the mean and standard deviation. In an analogous computation, with $\partial_\rho\mu = b_1(x_2-a_2)/b_2$

and $\partial_\rho \sigma = -\rho b_1/\sqrt{1-\rho^2}$, $\partial_\rho \phi(x_1|x_2; a, \Sigma)$ is expressed as

$$\begin{aligned}
&\frac{\partial}{\partial\rho}\phi_{\mu,\sigma}(x_1|x_2)\\
&= \frac{x_2-a_2}{\sqrt{2\pi}b_2\sqrt{1-\rho^2}}\left(\frac{x_1-\mu}{\sigma^2}\exp\left(-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right)1\{x>\mu\}\right.\\
&\quad\left. -\frac{\mu-x_1}{\sigma^2}\exp\left(-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right)1\{x<\mu\}\right) + \frac{\rho}{1-\rho^2}\left(\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right)\right.\\
&\quad\left. -\frac{1}{\sqrt{2\pi}\sigma}\left(\frac{x_1-\mu}{\sigma}\right)^2\exp\left(-\frac{1}{2}\left(\frac{x_1-\mu}{\sigma}\right)^2\right)\right)\\
&= (c_{1,\rho}(x_2)+c_{2,\rho})\left(\left(\frac{c_{1,\rho}(x_2)}{c_{1,\rho}(x_2)+c_{2,\rho}}r^+_{\mu,\sigma}(x_1|x_2) + \frac{c_{2,\rho}}{c_{1,\rho}(x_2)+c_{2,\rho}}\phi_{\mu,\sigma}(x_1|x_2)\right)\right.\\
&\quad\left. -\left(\frac{c_{1,\rho}(x_2)}{c_{1,\rho}(x_2)+c_{2,\rho}}r^-_{\mu,\sigma}(x_1|x_2) + \frac{c_{2,\rho}}{c_{1,\rho}(x_2)+c_{2,\rho}}m_{\mu,\sigma}(x_1|x_2)\right)\right). \qquad \text{(B.2)}
\end{aligned}$$

The pre-factors will have notation according to the parameter of study, namely for this parameter derivative

$$c_{1,\rho}(x_2) := \frac{x_2-a_2}{\sqrt{2\pi}b_2\sqrt{1-\rho^2}}, \quad \text{and} \quad c_{2,\rho} := \frac{\rho}{1-\rho^2}.$$

In Equation (B.2), we denote $r^+_{\mu,\sigma}$, and $r^-_{\mu,\sigma}$ as the density functions for the μ-shifted Rayleigh distributions on $[\mu,\infty)$, and $(-\infty,\mu]$ respectively, and $m_{\mu,\sigma}$ denotes the density function for the Double-Maxwell distribution. This follows the notation from Section 1.2.4.2, Equation (1.35). As each of the densities in the above equation are arguments x_1 predicated on the argument x_2, the generation of the random variables arise from first obtaining a draw from the normal random variable X_2. We denote the Rayleigh distributed random variables dependent on the normal random variable X_2 as $R^\pm|X_2$, specifically, $R^+|(X_2 = x_2) \sim r^+_{\mu(x_2),\sigma}$, and analogously for the negative variant. From Section 1.2.4.2, we let $H := H(0,1) = \sqrt{-2\ln(1-U)}$ be a standard Rayleigh distributed random variable, where $U \in (0,1)$ is a standard uniform random variable. The random variables $R^\pm|X_2$ are then an affine transformation of H, i.e., $R^\pm|X_2 = \mu(X_2) \pm \sigma H$. Additionally, let $W|X_2$ be a double-Maxwell random variable with mean $\mu(X_2)$ and scale parameter σ, particularly $W|(X_2 = x_2) \sim m_{\mu(x_2),\sigma}$. A standard Double-Maxwell random variable is generated via the acceptance-rejection method and the algorithm to generate this random variable is provided in [54]. Thus, with stochastic probabilities, $c_{1,\rho}(X_2)/(c_{1,\rho}(X_2)+c_{2,\rho})$ and

$c_{2,\rho}/(c_{1,\rho}(X_2)+c_{2,\rho})$, we define

$$\begin{pmatrix} X^+_{1,\rho} \\ X^-_{1,\rho} \end{pmatrix} = \frac{c_{1,\rho}(X_2)}{c_{1,\rho}(X_2)+c_{2,\rho}} \begin{pmatrix} R^+|X_2 \\ R^-|X_2 \end{pmatrix} + \frac{c_{2,\rho}}{c_{1,\rho}(X_2)+c_{2,\rho}} \begin{pmatrix} X_1|X_2 \\ W|X_2 \end{pmatrix}.$$

The measure-valued vectors are then denoted as $X^+_\rho = (X^+_{1,\rho}, X_2)^\mathsf{T}$ and X^-_ρ containing a similar structure. The corresponding measure valued derivative triple is written as $\left(c_{1,\rho}(X_2)+c_{2,\rho}, X^+_\rho, X^-_\rho\right)$.

For the parameter sensitivity w.r.t. the marginal mean, a_1, we note that this parameter is only within the mean μ of $\phi_{\mu,\sigma}$. As $\partial_{a_1}\mu = 1$, following (1.35)

$$\begin{aligned} \frac{\partial}{\partial a_1}\phi_{\mu,\sigma}(x_1|x_2) &= \frac{\partial}{\partial \mu}\phi_{\mu,\sigma}(x_1|x_2)\,\frac{\partial \mu}{\partial a_1} \\ &= \frac{1}{\sqrt{2\pi}b_1\sqrt{1-\rho^2}}\left(r^+_{\mu,\sigma}(x_1|x_2) - r^-_{\mu,\sigma}(x_1|x_2)\right). \end{aligned}$$

To generate this component of the measure-valued derivative, we require the same X_2-dependent μ-shifted Rayleigh random variables. Again, we denote $R^+|(X_2 = x_2) \sim r^+_{\mu(x_2),\sigma}$, and similarly for the negative variant, and we define the measure-valued derivative random variables $X^\pm_{1,a_1}$ as $X^\pm_{1,a_1} = R^\pm|X_2$. As before, the measure-valued derivative random vectors are defined by $X^+_{a_1} = (X^+_{1,a_1}, X_2)^\mathsf{T}$, similarly for $X^-_{a_1}$. The measure-valued derivative triple for this derivative is then $\left(c_{a_1}, X^+_{a_1}, X^+_{a_1}\right)$, denoting $c_{a_1} = c_\mu$ as the pre-factor from (1.35).

The derivative w.r.t. the marginal standard deviation, b_1, requires the chain rule, Equation (1.37), attaining a corresponding result to the sensitivity w.r.t. ρ. As $\partial_{b_1}\mu = \rho(x_2-a_2)/b_2$, and $\partial_{b_1}\sigma = \sqrt{1-\rho^2}$:

$$\begin{aligned} \frac{\partial}{\partial b_1}\phi_{\mu,\sigma}(x_1|x_2) &= \frac{\partial}{\partial \mu}\phi_{\mu,\sigma}(x_1|x_2)\frac{\partial \mu}{\partial b_1} + \frac{\partial}{\partial \sigma}\phi_{\mu,\sigma}(x_1|x_2)\frac{\partial \sigma}{\partial b_1} \\ &= \frac{\rho(x_2-a_2)}{\sqrt{2\pi}b_1 b_2\sqrt{1-\rho^2}}\left(r^+_{\mu,\sigma}(x_1|x_2) - r^-_{\mu,\sigma}(x_1|x_2)\right) \\ &\quad + \frac{1}{b_1}\left(m_{\mu,\sigma}(x_1|x_2) - \phi_{\mu,\sigma}(x_1|x_2)\right). \end{aligned}$$

The corresponding pre-factors for the derivative w.r.t. μ and σ are

$$c_{1,b_1}(x_2) := \frac{\rho(x_2-a_2)}{\sqrt{2\pi}b_1 b_2\sqrt{1-\rho^2}}, \quad \text{and} \quad c_{2,b_1} := \frac{1}{b_1},$$

The pre-factor c_{1,b_1} is also a function of the argument x_2.

Generation of the random variables X^+_{1,b_1}, X^-_{1,b_1} follows from the same precepts as for the derivative w.r.t. ρ. We utilize the same notation as earlier, specifically, $R^\pm|X_2$ to denote the μ-shifted Rayleigh random variables, and $W|X_2$ to denote the double-Maxwell random variable in which μ is dependent on X_2. We denote that $c_{1,b_1}(X_2)/(c_{1,b_1}(X_2)+c_{2,b_1})$ and $c_{1,b_1}(X_2)/(c_{1,b_1}(X_2)+c_{2,b_1})$ are probabilities, functions of the normal random variable X_2. The random variables X^+_{1,b_1}, X^-_{1,b_1} for the derivative w.r.t. the parameter b_1 is

$$\begin{pmatrix} X^+_{1,b_1} \\ X^-_{1,b_1} \end{pmatrix} = \frac{c_{1,b_1}(X_2)}{c_{1,b_1}(X_2)+c_{2,b_1}} \begin{pmatrix} R^+|X_2 \\ R^-|X_2 \end{pmatrix} + \frac{c_{2,b_1}}{c_{1,b_1}(X_2)+c_{2,b_1}} \begin{pmatrix} W|X_2 \\ X_1|X_2 \end{pmatrix}.$$

As before, the second element of the measure-valued derivative random vectors $X^\pm_{b_1} = (X^\pm_{1,b_1}, X_2)^\top$ is unchanged. The corresponding triple is then written as $\left(c_{1,b_1}(X_2)+c_{2,b_1}, X^+_{b_1}, X^-_{b_1}\right)$.

B.2 Alternative Representation

For the parameter derivatives w.r.t. to the common correlation ρ, and a marginal standard deviation, both derivatives contain a stochastic pre-factor in their measure-valued derivatives, specifically containing the standardization $(x_2-a_2)/b_2$. Combining this term with the normal density $\phi_{\mu,\sigma}(x_2)$ yields an additional pair of Rayleigh density functions in the different measure-valued derivative. These Rayleigh density functions are denoted by $s^\pm_{a_2,b_2}(x_2)$. Since this stochastic pre-factor is not present in the sensitivity w.r.t. the marginal mean parameter a_1, there is no alternative representation of this form for this parameter.

With regard to the sensitivity of the parameter ρ, the rewriting of Equation (B.2) to include the a_2-shifted Rayleigh random variables is given below:

$$\begin{aligned}
&\frac{\partial}{\partial\rho}\phi_{a,\Sigma}(x) \\
&= \frac{x_2-a_2}{\sqrt{2\pi}b_2\sqrt{1-\rho^2}}\left(r^+_{\mu,\sigma}(x_1|x_2)-r^-_{\mu,\sigma}(x_1|x_2)\right)\cdot\frac{1}{\sqrt{2\pi}b_2}\exp\left(-\frac{1}{2}\left(\frac{x_2-a_2}{b_2}\right)^2\right) \\
&\quad+\frac{\rho}{1-\rho^2}\left(m_{\mu,\sigma}(x_1|x_2)-\phi_{\mu,\sigma}(x_1|x_2)\right)\phi_{a_2,b_2}(x_2)
\end{aligned}$$

$$= \frac{1}{2\pi\sqrt{1-\rho^2}}\Big(r^+_{\mu,\sigma}(x_1|x_2) - r^-_{\mu,\sigma}(x_1|x_2)\Big)$$
$$\cdot\left(\frac{x_2-a_2}{b_2^2}\exp\left(-\frac{1}{2}\left(\frac{x_2-a_2}{b_2}\right)^2\right)1\{x_2>a_2\} - \frac{a_2-x_2}{b_2^2}\exp\left(-\frac{1}{2}\left(\frac{x_2-a_2}{b_2}\right)^2\right)1\{x_2<a_2\}\right)$$
$$+ \frac{\rho}{1-\rho^2}\big(m_{\mu,\sigma}(x_1|x_2) - \phi_{\mu,\sigma}(x_1|x_2)\big)\phi_{a_2,b_2}(x_2)$$
$$:= \frac{1}{2\pi\sqrt{1-\rho^2}}\Big(r^+_{\mu,\sigma}(x_1|x_2) - r^-_{\mu,\sigma}(x_1|x_2)\Big)\Big(s^+_{a_2,b_2}(x_2) - s^-_{a_2,b_2}(x_2)\Big)$$
$$+ \frac{\rho}{1-\rho^2}\big(\phi_{\mu,\sigma}(x_1|x_2) - m_{\mu,\sigma}(x_1|x_2)\big)\phi_{a_2,b_2}(x_2).$$

We label $d_{1,\rho} = 1/(2\pi\sqrt{1-\rho^2})$ and $d_{2,\rho} = c_{2,\rho}$ as the pre-factors of this modified expression.

For this representation, we denote the measure-valued derivative random vectors by $Y^\pm_\rho$. In terms of a conditional form of the bivariate density, the density functions associated with $Y^\pm_\rho$ are

$$\frac{\partial}{\partial\rho}\phi_{a,\Sigma}(x)$$
$$= (2d_{1,\rho} + d_{2,\rho})$$
$$\cdot\Bigg(\bigg(\frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}} r^+_{\mu,\sigma}(x_1|x_2)s^+_{a_2,b_2}(x_2) + \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}} r^-_{\mu,\sigma}(x_1|x_2)s^-_{a_2,b_2}(x_2)$$
$$+ \frac{d_{2,\rho}}{2d_{1,\rho}+d_{2,\rho}}\phi_{\mu,\sigma}(x_2|x_1)\phi_{a_2,b_2}(x_2)\bigg) - \bigg(\frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}} r^+_{\mu,\sigma}(x_1|x_2)s^-_{a_2,b_2}(x_2)$$
$$+ \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}} r^-_{\mu,\sigma}(x_1|x_2)s^+_{a_2,b_2}(x_2) + \frac{d_{2,\rho}}{2d_{1,\rho}+d_{2,\rho}} m_{\mu,\sigma}(x_2|x_1)\phi_{a_2,b_2}(x_2)\bigg)\Bigg).$$

We define $I := I(0,1) = \sqrt{-2\ln(1-V)}$ as an independent copy of the standard Rayleigh variable H, where $V \in (0,1)$ is a standard uniform random variable. Let $S^\pm \sim s^\pm_{a_2,b_2}$, represent the random variables of the a_2-shifted Rayleigh distributions. The affine relationship that relates $S^\pm$ to the random variable I is $S^\pm = a_2 \pm b_2 I$.

In the make up of the random vectors $Y^\pm_\rho$, we denote by $R^\pm|S^+$, with probability $d_{1,\rho}/(2d_{1,\rho}+d_{2,\rho})$, as the random variable generated via the conditional density function $r^\pm_{\mu,\sigma}(x_1|x_2)$ given that the mean $\mu(S^+) = a_1 + \rho b_1(S^+ - a_2)/b_2$. Specifically, the distribution of random variable pair is of the form

$$\begin{pmatrix} R^\pm|(S^+ = x_2) \\ S^+ \end{pmatrix} \sim \begin{pmatrix} r^\pm_{\mu(x_2),\sigma}(x_1|x_2) \\ s^+_{a_2,b_2}(x_2). \end{pmatrix}$$

An analogous result holds for the distribution of the vector $(R^{\pm}|(S^{-}=x_2), S^{-})^{\mathsf{T}}$. With remaining probability $d_{2,\rho}/(2d_{1,\rho}+d_{2,\rho})$, Y_ρ^+ is distributed as earlier, via the conditional Double-Maxwell density function $m_{\mu,\sigma}(x_1|x_2)$ in which mean μ is a function of the normal random variable X_2. The form of the measure-valued derivative random variables are then

$$Y_\rho^+ = \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} R^+|S^+ \\ S^+ \end{pmatrix} + \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} R^-|S^- \\ S^- \end{pmatrix} + \frac{d_{2,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} X_1|X_2 \\ X_2, \end{pmatrix},$$

and

$$Y_\rho^- = \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} R^+|S^- \\ S^- \end{pmatrix} + \frac{d_{1,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} R^-|S^+ \\ S^+ \end{pmatrix} + \frac{d_{2,\rho}}{2d_{1,\rho}+d_{2,\rho}}\begin{pmatrix} W|X_2 \\ X_2 \end{pmatrix}.$$

The alternative measure-valued derivative triple for X w.r.t. the common correlation is $\left(2d_{1,\rho}+d_{2,\rho}, Y_\rho^+, Y_\rho^-\right)$.

For the parameter derivative $\partial_{b_1}\phi_{a,\Sigma}$, a similar modification yields

$$\begin{aligned}
&\frac{\partial}{\partial b_1}\phi_{a,\Sigma}(x)\\
&= \frac{\rho}{2\pi b_1\sqrt{1-\rho^2}}\left(r^+_{\mu,\sigma}(x_1|x_2) - r^-_{\mu,\sigma}(x_1|x_2)\right)\left(s^+_{a_2,b_2}(x_2) - s^-_{a_2,b_2}(x_2)\right)\\
&\quad + \frac{1}{b_1}\left(m_{\mu,\sigma}(x_1|x_2) - \phi_{\mu,\sigma}(x_1|x_2)\right)\phi_{a_2,b_2}(x_2)\\
&= (2d_{1,b_1}+d_{2,b_1})\\
&\quad\cdot\Bigg(\bigg(\frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}r^+_{\mu,\sigma}(x_1|x_2)s^+_{a_2,b_2}(x_2) + \frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}r^-_{\mu,\sigma}(x_1|x_2)s^-_{a_2,b_2}(x_2)\\
&\quad + \frac{d_{2,b_1}}{2d_{1,b_1}+d_{2,b_1}}m_{\mu,\sigma}(x_2|x_1)\phi_{a_2,b_2}(x_2)\bigg) - \bigg(\frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}r^+_{\mu,\sigma}(x_1|x_2)s^-_{a_2,b_2}(x_2)\\
&\quad + \frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}r^-_{\mu,\sigma}(x_1|x_2)s^+_{a_2,b_2}(x_2) + \frac{d_{2,b_1}}{2d_{1,b_1}+d_{2,b_1}}\phi_{\mu,\sigma}(x_2|x_1)\phi_{a_2,b_2}(x_2)\bigg)\Bigg).
\end{aligned}$$

The pre-factors are denoted by $d_{1,b_1} = \rho/(2\pi b_1\sqrt{1-\rho^2})$ and $d_{2,b_1} = c_{2,b_1}$. Condequently, the measure-valued derivative random variables $Y^+_{b_1}$ and $Y^-_{b_1}$ share an akin form to the measure-valued derivative random variables $Y^{\pm}_\rho$, and we use the same notation to represent each of the random variables. Translating the measure-valued derivative of $\partial_{b_1}\phi_{a,\Sigma}$ into random vectors, we obtain

$$Y^+_{b_1} = \frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}\begin{pmatrix} R^+|S^+ \\ S^+ \end{pmatrix} + \frac{d_{1,b_1}}{2d_{1,b_1}+d_{2,b_1}}\begin{pmatrix} R^-|S^- \\ S^- \end{pmatrix} + \frac{d_{2,b_1}}{2d_{1,b_1}+d_{2,b_1}}\begin{pmatrix} W|X_2 \\ X_2, \end{pmatrix}$$

and

$$Y_{b_1}^{-} = \frac{d_{1,b_1}}{2d_{1,b_1} + d_{2,b_1}} \begin{pmatrix} R^{+}|S^{-} \\ S^{-} \end{pmatrix} + \frac{d_{1,b_1}}{2d_{1,b_1} + d_{2,b_1}} \begin{pmatrix} R^{-}|S^{+} \\ S^{+} \end{pmatrix} + \frac{d_{2,b_1}}{2d_{1,b_1} + d_{2,b_1}} \begin{pmatrix} X_1|X_2 \\ X_2 \end{pmatrix}.$$

The measure-valued derivative triple for the sensitivity for the bivariate normal random variable X w.r.t. the marginal standard deviation b_1 is then $\left(2d_{1,b_1} + d_{2,b_1}, Y_{b_1}^{+}, Y_{b_2}^{-}\right)$.