Appendix A

Derivations of Chapter 4

In this appendix, we elaborate how we calculate Table 4.1 and Proposition 4.2 of Chapter 4. According to Equation (4.3), all agents maximize their equilibrium payoffs in π^0 by choosing proposals at the proposing stage that maximize their net excesses. Due to the symmetry of agents 1 and 2, we assume that $v_1^0 = v_2^0 = v$ and we denote v_3^0 as v_3. Then we obtain the net excesses,

$$\begin{aligned} e^0(12) &= \tfrac{2}{3}(1+\varepsilon)\delta - 2\delta v, \\ e^0(13) &= \tfrac{1}{3}(3+2\varepsilon)\delta + (1-\delta) - \delta v - \delta v_3, \\ e^0(23) &= \tfrac{1}{3}(3+2\varepsilon)\delta + (1-\delta) - \delta v - \delta v_3, \\ e^0(N) &= 1+\varepsilon - 2\delta v - \delta v_3, \end{aligned}$$

and obviously we have $e^0(13) = e^0(23)$. It turns out that we only need to compare $e^0(12), e^0(13)$ and $e^0(N)$. The idea is that we first compare $e^0(12)$, $e^0(13)$ and $e^0(N)$ before we calculate the equilibrium payoffs. Given the ordering, we can obtain the optimal proposals and are able to calculate the equilibrium payoffs. Finally, we check for the conditions that support the ordering of $e^0(12)$, $e^0(13)$ and $e^0(N)$. We start from agent 3 and distinguish the following two cases,

$$e^0(13) \geq e^0(N), \tag{A.1}$$
$$e^0(13) < e^0(N). \tag{A.2}$$

Next we identify subcases by inserting the additional term $e^0(12)$ at every possible position into the above two inequalities. Note that the equality should also get attention since it may result in mixed strategies.

A.0.3 Inequality (A.1)

There are three possibilities in this situation,

$$e^0(12) \geq e^0(13) \geq e^0(N), \tag{A.3}$$
$$e^0(13) > e^0(12) \geq e^0(N), \tag{A.4}$$
$$e^0(13) \geq e^0(N) > e^0(12). \tag{A.5}$$

A.0.3.1 Inequality (A.3)

In terms of inequality (A.3), we have the following four subcases,

$$e^0(12) = e^0(13) = e^0(N), \tag{A.6}$$
$$e^0(12) = e^0(13) > e^0(N), \tag{A.7}$$
$$e^0(12) > e^0(13) = e^0(N), \tag{A.8}$$
$$e^0(12) > e^0(13) > e^0(N). \tag{A.9}$$

Equation (A.6) corresponds to the case in which all agents mix over all possible proposals, which is called Region E. Inequality (A.7) corresponds to the case that agent 1 mixes over $\{1, 2\}$ and $\{1, 3\}$, agent 2 mixes over $\{1, 2\}$ and $\{2, 3\}$, agent 3 mixes over $\{1, 3\}$ and $\{2, 3\}$ with equal probabilities $\frac{1}{2}$. And this is named Region D. Inequality (A.8) corresponds to the case that agents 1 and 2 propose $\{1, 2\}$ and agent 3 mixes over $\{1, 3\}$, $\{2, 3\}$ and the grand coalition. Inequality (A.9) corresponds to the case that agents 1 and 2 propose $\{1, 2\}$ and agent 3 mixes over $\{1, 3\}$ and $\{2, 3\}$ with equal probabilities $\frac{1}{2}$. In Section A.2, we will show that inequalities (A.8) and (A.9) cannot support an equilibrium.

A.0.3.2 Inequality (A.4)

In this situation, agent 1 proposes $\{1, 3\}$, agent 2 proposes $\{2, 3\}$ and agent 3 mixes over $\{1, 3\}$ and $\{2, 3\}$ with equal probabilities $\frac{1}{2}$. And this is called Region B.

A.0.3.3 Inequality (A.5)

We have the following two situations

$$e^0(13) > e^0(N) > e^0(12), \tag{A.10}$$
$$e^0(13) = e^0(N) > e^0(12). \tag{A.11}$$

Inequality (A.10) corresponds to Region B again. Inequality (A.11) corresponds to the situation that agent 1 mixes over $\{1, 3\}$ and the grand coalition, agent 2 mixes over $\{2, 3\}$ and the grand coalition, and agent 3 mixes over $\{1, 3\}$, $\{2, 3\}$ and the grand coalition. This is named Region C.

A.0.4 Inequality (A.2)

There are three possibilities as well,

$$e^0(12) \geq e^0(N) > e^0(13), \tag{A.12}$$
$$e^0(N) > e^0(12) \geq e^0(13), \tag{A.13}$$
$$e^0(N) > e^0(13) > e^0(12). \tag{A.14}$$

A.0.4.1 Inequalities (A.13) and (A.14)

Inequalities (A.13) and (A.14) correspond to the situation that every agent proposes the grand coalition which is called Region A.

A.0.4.2 Inequality (A.12)

With respect to inequality (A.12), there are following two cases,

$$e^0(12) = e^0(N) > e^0(13), \tag{A.15}$$
$$e^0(12) > e^0(N) > e^0(13). \tag{A.16}$$

Inequality (A.15) corresponds to the case that agents 1 and 2 mix over $\{1,2\}$ and the grand coalition, agent 3 proposes the grand coalition. Inequality (A.16) corresponds to the case that agents 1 and 2 propose $\{1,2\}$ and agent 3 proposes the grand coalition. In Section A.2, we will show that inequalities (A.15) and (A.16) cannot support an equilibrium.

In Table A.1, we summarize the above discussions. We will derive regions A, B, C, D and E separately in Section A.1 and put those no equilibrium cases in Section A.2.

A.1 Equilibrium regions

A.1.1 Region A

When $e^0(N) > \max\{e^0(12),\, e^0(13)\}$, it is optimal for all agents to propose the grand coalition immediately. For the equilibrium payoffs, we have

$$v = \tfrac{1}{3}(1 + \varepsilon - 2\delta v - \delta v_3) + \delta v,$$
$$v_3 = \tfrac{1}{3}(1 + \varepsilon - 2\delta v - \delta v_3) + \delta v_3.$$

Solving the above two equations, we obtain $v = v_3 = \frac{1+\varepsilon}{3}$. After checking the condition $e^0(N) > \max\{e^0(12),\, e^0(13)\}$, we obtain $\varepsilon > \frac{\delta}{3(1-\delta)}$.

A.1.2 Region B

When $e^0(13) > \max\{e^0(12),\, e^0(N)\}$, agent 1 proposes $\{1,\,3\}$, agent 2 proposes $\{2,3\}$ and agent 3 mixes over $\{1,3\}$ and $\{2,3\}$ with equal probabilities $\frac{1}{2}$. For the equilibrium payoffs, we have

$$v = \tfrac{1}{3}\left[\frac{3+2\varepsilon}{3}\delta + (1-\delta) - \delta v - \delta v_3\right] + \tfrac{1}{2}\delta v + \tfrac{1}{2}\delta\frac{\varepsilon}{3},$$
$$v_3 = \tfrac{1}{3}\left[\frac{3+2\varepsilon}{3}\delta + (1-\delta) - \delta v - \delta v_3\right] + \delta\frac{1+\varepsilon}{3}.$$

Solving the above two equations, we obtain

$$v = \frac{2(1-\delta)(2\delta\varepsilon+3) + \delta\varepsilon(3-2\delta)}{3(6-5\delta)},$$
$$v_3 = \frac{(2-\delta)(2\delta\varepsilon+3) - \delta^2\varepsilon}{3(6-5\delta)}.$$

After checking the condition $e^0(13) > \max\{e^0(12),\, e^0(N)\}$, we obtain $\frac{(6-7\delta)(\delta-3)}{3\delta^2(1-\delta)} < \varepsilon < \frac{2\delta}{6-3\delta-2\delta^2}$.

Table A.1: All possible equilibria and their regions in the river game, where - means this cannot occur in equilibrium

possible orderings	proposing strategies	region
$e^0(12) = e^0(13) = e^0(N)$	Every agent mixes over all possible proposals.	E
$e^0(12) = e^0(13) > e^0(N)$	$1 \rightarrow \{\{1,2\},\{1,3\}\}$, $2 \rightarrow \{\{1,2\},\{2,3\}\}$, $3 \rightarrow \{\{1,3\},\{2,3\}\}$	D
$e^0(12) > e^0(13) = e^0(N)$	$1,2 \rightarrow \{1,2\}$, $3 \rightarrow \{\{1,3\},\{2,3\},N\}$	-
$e^0(12) > e^0(13) > e^0(N)$	$1,2 \rightarrow \{1,2\}$, $3 \rightarrow \{\{1,3\},\{2,3\}\}$	-
$e^0(13) > e^0(12) \geq e^0(N)$, $e^0(13) > e^0(N) > e^0(12)$	$1 \rightarrow \{1,3\}$, $2 \rightarrow \{2,3\}$, $3 \rightarrow \{\{1,3\},\{2,3\}\}$	B
$e^0(13) = e^0(N) > e^0(12)$	$1 \rightarrow \{\{1,3\},N\}$, $2 \rightarrow \{\{2,3\},N\}$, $3 \rightarrow \{\{1,3\},\{2,3\},N\}$	C
$e^0(12) = e^0(N) > e^0(13)$	$1 \rightarrow \{\{1,2\},N\}$, $2 \rightarrow \{\{1,2\},N\}$, $3 \rightarrow N$	-
$e^0(12) > e^0(N) > e^0(13)$	$1 \rightarrow \{1,2\}$, $2 \rightarrow \{1,2\}$, $3 \rightarrow N$	-
$e^0(N) > e^0(12) \geq e^0(13)$, $e^0(N) > e^0(13) > e^0(12)$	Everyone proposes the grand coalition.	A

A.1.3 Region C

When $e^0(13) = e^0(N) > e^0(12)$, agent 1 mixes over $\{1,3\}$ and the grand coalition, agent 2 mixes over $\{2,3\}$ and the grand coalition, and agent 3 mixes over $\{1,3\}$, $\{2,3\}$ and the grand coalition. It follows directly from the condition $e^0(13) = e^0(N) > e^0(12)$ that,

$$v = \frac{3-2\delta}{3\delta}\varepsilon, \tag{A.17}$$
$$\delta v_3 < \tfrac{1}{3}(1+\varepsilon)(3-2\delta). \tag{A.18}$$

For the mixed strategies, the following equations hold

$$\sigma_1(13) + \sigma_1(N) = 1,$$
$$\sigma_2(23) + \sigma_2(N) = 1,$$
$$\sigma_3(13) + \sigma_3(23) + \sigma_3(N) = 1.$$

For the equilibrium payoffs, we have

$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v[1+\sigma_2(N)+\sigma_3(13)+\sigma_3(N)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_2(23)+\sigma_3(23)], \tag{A.19}$$
$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v[1+\sigma_1(N)+\sigma_3(23)+\sigma_3(N)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_1(13)+\sigma_3(13)], \tag{A.20}$$
$$v_3 = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \delta v_3. \tag{A.21}$$

From Equation (A.19),

$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v[1+\sigma_2(N)+\sigma_3(13)+\sigma_3(N)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[1-\sigma_2(N)+1-\sigma_3(13)-\sigma_3(N)].$$

From Equation (A.20),

$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v[1+\sigma_1(N)+\sigma_3(23)+\sigma_3(N)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[1-\sigma_1(N)+1-\sigma_3(23)-\sigma_3(N)].$$

Since agents 1 and 2 are symmetric, i.e., equations (A.19) and (A.20) are equal, we have $\sigma_1(N) + \sigma_3(23) = \sigma_2(N) + \sigma_3(13)$. From Equation (A.21), we immediately get that

$$v_3 = \frac{3-3\varepsilon+4\varepsilon\delta}{3(3-2\delta)}. \tag{A.22}$$

From condition (A.18), we get that $\varepsilon > \frac{(3-\delta)(4\delta-3)}{9(1-\delta)}$. We rewrite the symmetry condition as $x = \sigma_2(N) - \sigma_3(23) = \sigma_1(N) - \sigma_3(13)$, then the payoff function of agent 1 can be written in terms of x as

$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v(2+x) + \frac{\delta\varepsilon}{9}(1-x).$$

Plugging (A.17) and (A.22) into the above equation, we obtain $x = \frac{9\varepsilon-3\delta-6\delta\varepsilon-2\delta^2\varepsilon}{\delta\varepsilon(3-2\delta)}$. Now we are going to consider the range of x. Obviously, we have $x \leq 1$,[1] and $x \geq -\frac{1}{2}$.[2] First consider that $x \leq 1$, we obtain that $\frac{9\varepsilon-3\delta-6\delta\varepsilon-2\delta^2\varepsilon}{\delta\varepsilon(3-2\delta)} \leq 1$, and by further simplification, we get that $\varepsilon \leq \frac{\delta}{3(1-\delta)}$. Then we consider that $x \geq -\frac{1}{2}$, we obtain that $\frac{9\varepsilon-3\delta-6\delta\varepsilon-2\delta^2\varepsilon}{\delta\varepsilon(3-2\delta)} \geq -\frac{1}{2}$, and we have $\varepsilon \geq \frac{2\delta}{6-3\delta-2\delta^2}$. In conclusion, the ranges of δ and ε in Region C are given by the following three inequalities,

$$\varepsilon > \frac{(3-\delta)(4\delta-3)}{9(1-\delta)},\ \varepsilon \leq \frac{\delta}{3(1-\delta)} \text{ and } \varepsilon \geq \frac{2\delta}{6-3\delta-2\delta^2}.$$

A.1.4 Region D

When $e^0(12) = e^0(13) > e^0(N)$, agent 1 mixes over $\{1,2\}$ and $\{1,3\}$, agent 2 mixes over $\{1,2\}$ and $\{2,3\}$, and agent 3 mixes over $\{1,3\}$ and $\{2,3\}$. For the mixed strategies, the following equations hold

$$\begin{aligned}\sigma_1(12) + \sigma_1(13) &= 1,\\ \sigma_2(12) + \sigma_2(23) &= 1,\\ \sigma_3(13) + \sigma_3(23) &= 1.\end{aligned}$$

For the equilibrium payoff functions, we have

$$v = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)}{3}\delta - 2\delta v\right] + \tfrac{1}{3}\delta v[1+\sigma_2(12)+\sigma_3(13)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_2(23)+\sigma_3(23)], \qquad \text{(A.23)}$$

$$v = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)}{3}\delta - 2\delta v\right] + \tfrac{1}{3}\delta v[1+\sigma_1(12)+\sigma_3(23)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_1(13)+\sigma_3(13)], \qquad \text{(A.24)}$$

$$v_3 = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)}{3}\delta - 2\delta v\right] + \tfrac{1}{3}\delta v_3[1+\sigma_1(13)+\sigma_2(23)] + \tfrac{1}{3}\delta\frac{1+\varepsilon}{3}[\sigma_1(12)+\sigma_2(12)].$$

For Equation (A.23), we rewrite as

$$v = \tfrac{1}{3}\left[\tfrac{2(1+\varepsilon)}{3}\delta - 2\delta v\right] + \tfrac{1}{3}\delta v[1+\sigma_2(12)+\sigma_3(13)] + \tfrac{1}{3}\delta\tfrac{\varepsilon}{3}[1-\sigma_2(12)+1-\sigma_3(13)].$$

For Equation (A.24), we rewrite as

$$v = \tfrac{1}{3}\left[\tfrac{2(1+\varepsilon)}{3}\delta - 2\delta v\right] + \tfrac{1}{3}\delta v[1+\sigma_1(12)+\sigma_3(23)] + \tfrac{1}{3}\delta\tfrac{\varepsilon}{3}[1-\sigma_1(12)+1-\sigma_3(23)].$$

Since agents 1 and 2 are symmetric, we have $\sigma_1(12)+\sigma_3(23) = \sigma_2(12)+\sigma_3(13)$. Combining the mixed strategies equations together, we express every σ in terms of $\sigma_1(13)$ and $\sigma_2(23)$

[1] We have $x = 1$ when $\sigma_2(N) = \sigma_1(N) = 1, \sigma_3(23) = \sigma_3(13) = 0$. And this boundary divides Region A and C.

[2] To see this, note that $2x = \sigma_1(N) + \sigma_2(N) - \sigma_3(13) - \sigma_3(23) \geq -1$. This corresponds to the left boundary of Region B. Indeed, when $2x = -1$, we end up with the proposing strategies in Region B.

and obtain,

$$\begin{aligned}\sigma_1(12) &= 1-\sigma_1(13),\\ \sigma_2(12) &= 1-\sigma_2(23),\\ \sigma_3(23) &= \frac{\sigma_1(13)+1-\sigma_2(23)}{2},\\ \sigma_3(13) &= \frac{\sigma_2(23)+1-\sigma_1(13)}{2}.\end{aligned}$$

Furthermore, we assume that $\sigma_1(13)+\sigma_2(23)=x$ and $x\in[0,2]$. Then we can write down the payoff functions in terms of x,

$$\begin{aligned}v &= \tfrac{1}{3}\left[\frac{2(1+\varepsilon)}{3}\delta-2\delta v\right]+\tfrac{1}{3}\delta v\frac{5-x}{2}+\frac{\delta\varepsilon(1+x)}{18},\\ v_3 &= \tfrac{1}{3}\left[\frac{2(1+\varepsilon)}{3}\delta-2\delta v\right]+\tfrac{1}{3}\delta v_3(1+x)+\frac{(1+\varepsilon)\delta(2-x)}{9}.\end{aligned}$$

From the payoff functions, we obtain

$$v=\frac{2(1+\varepsilon)\delta+\delta\varepsilon\frac{x+1}{2}}{9-3\delta\frac{1-x}{2}}, \tag{A.25}$$

$$v_3=\frac{2(1+\varepsilon)\delta-6\delta v+\delta(1+\varepsilon)(2-x)}{9-3\delta-3\delta x}. \tag{A.26}$$

Still we need to consider the condition $e^0(12)=e^0(13)>e^0(N)$,

$$\delta v_3=\delta v+1-\tfrac{2}{3}\delta, \tag{A.27}$$

$$\delta v>\varepsilon-\tfrac{2}{3}\delta\varepsilon. \tag{A.28}$$

Plugging equations (A.25) and (A.26) into the condition (A.27), we obtain

$$x=\frac{1}{-2\delta(1-\delta)}(3-4\delta+3\delta^2-3\delta\varepsilon+3\delta^2\varepsilon-\sqrt{\Delta}) \tag{A.29}$$

where $\Delta=81-180\delta+130\delta^2-28\delta^3+\delta^4-18\delta\varepsilon+30\delta^2\varepsilon-18\delta^3\varepsilon+6\delta^4\varepsilon+9\delta^2\varepsilon^2-18\delta^3\varepsilon^2+9\delta^4\varepsilon^2$.

A.1.4.1 Discussion of the constraint that $x\in[0,2]$

First, for $\delta\in(0,1)$ and $\varepsilon\in[0,1]$, we always have $x>0$. For $x\le 2$, we obtain

$$3-\delta^2-3\delta\varepsilon+3\delta^2\varepsilon\ge\sqrt{\Delta}. \tag{A.30}$$

The left-hand side of (A.30) for $\delta\in(0,1)$ and $\varepsilon\in[0,1]$ is positive,[3] so we can take the square of both sides and obtain,

$$-3\delta^4\varepsilon+6\delta^3\varepsilon-3\delta^2\varepsilon+7\delta^3-34\delta^2+45\delta-18\ge 0$$

and by further simplification, we get $\varepsilon\le\frac{(6-7\delta)(\delta-3)}{3\delta^2(1-\delta)}$, which is exactly the right boundary of Region B.

[3]This can be shown by simulation.

A.1.4.2 Discussion of the equilibrium condition $\delta v > \varepsilon - \frac{2}{3}\delta\varepsilon$

Plugging (A.25) into $\delta v > \varepsilon - \frac{2}{3}\delta\varepsilon$, we obtain $\delta\frac{2(1+\varepsilon)\delta+\delta\varepsilon\frac{1+x}{2}}{9-3\delta\frac{1-x}{2}} > \varepsilon - \frac{2}{3}\delta\varepsilon$. and by further simplification, we get

$$x < \frac{9\varepsilon - \frac{15\delta\varepsilon}{2} - 2\delta^2 - \frac{3}{2}\varepsilon\delta^2}{\frac{3}{2}\delta\varepsilon(\delta-1)}. \tag{A.31}$$

Combining this expression with the expression (A.29), we have

$$8\delta^2 + 18\delta\varepsilon - 27\varepsilon + 15\delta^2\varepsilon - 9\delta\varepsilon^2 + 9\delta^2\varepsilon^2 > 3\varepsilon\sqrt{\Delta}. \tag{A.32}$$

Taking the square of both sides,[4] the left-hand side of the inequality (A.32) reads

$$(81\delta^4 - 162\delta^3 + 81\delta^2)\varepsilon^4 + (270\delta^4 + 54\delta^3 - 810\delta^2 + 486\delta)\varepsilon^3$$
$$+(369\delta^4 + 396\delta^3 - 486\delta^2 - 972\delta + 729)\varepsilon^2 + (240\delta^4 + 288\delta^3 - 432\delta^2)\varepsilon + 64\delta^4$$

The right-hand side of the inequality (A.32) reads

$$(81\delta^4 - 162\delta^3 + 81\delta^2)\varepsilon^4 + (54\delta^4 - 162\delta^3 + 270\delta^2 - 162\delta)\varepsilon^3 + (9\delta^4 - 252\delta^3 + 1170\delta^2 - 1620\delta + 729)\varepsilon^2.$$

By further simplification of (A.32), we get

$$(27\delta^3 + 27\delta^2 - 135\delta + 81)\varepsilon^3 + (45\delta^3 + 81\delta^2 - 207\delta + 81)\varepsilon^2 + (30\delta^3 + 36\delta^2 - 54\delta)\varepsilon + 8\delta^3 > 0.$$

First, we set the above inequality to zero and get the following three roots

$$\varepsilon_1 = \frac{2\delta}{3(1-\delta)},$$
$$\varepsilon_2 = \frac{9 - 8\delta - 3\delta^2 - \sqrt{81 - 144\delta + 58\delta^2 + 16\delta^3 - 7\delta^4}}{6(\delta+3)(\delta-1)},$$
$$\varepsilon_3 = \frac{9 - 8\delta - 3\delta^2 + \sqrt{81 - 144\delta + 58\delta^2 + 16\delta^3 - 7\delta^4}}{6(\delta+3)(\delta-1)}.$$

For $\delta \in [0,1)$, we have $\varepsilon_3 \leq 0$. In Figure A.1, we plot ε_1 and ε_2. Obviously for $\delta \in [0,1)$, we have $\varepsilon_1 \geq \varepsilon_2 \geq 0 \geq \varepsilon_3$. Hence, we obtain the following solution $\varepsilon > \varepsilon_1$, or $\varepsilon_3 < \varepsilon < \varepsilon_2$. Moreover, given that $\varepsilon_3 \leq 0$ for $\delta \in [0,1)$, we get $\varepsilon > \varepsilon_1$, or $0 \leq \varepsilon < \varepsilon_2$.

A.1.4.3 Conclusion

Combining the results of Section A.1.4.1 and A.1.4.2, we obtain $0 < \delta < 1$, $0 \leq \varepsilon < \varepsilon_2$, and $\varepsilon \leq \frac{(6-7\delta)(\delta-3)}{3\delta^2(1-\delta)}$. In Figure A.2, Region D is below both curves[5] and above the horizontal axis.

[4] Simulation for δ and ε satisfying the constraint $\varepsilon \leq \frac{18-27\delta+7\delta^2}{3(-1+\delta)\delta^2}$ indicates that the right-hand side of (A.32) is positive. So we can take the square of both sides for (A.32).

[5] The solid line denotes ε_2 and the dotted line stands for $\frac{(6-7\delta)(\delta-3)}{3\delta^2(1-\delta)}$.

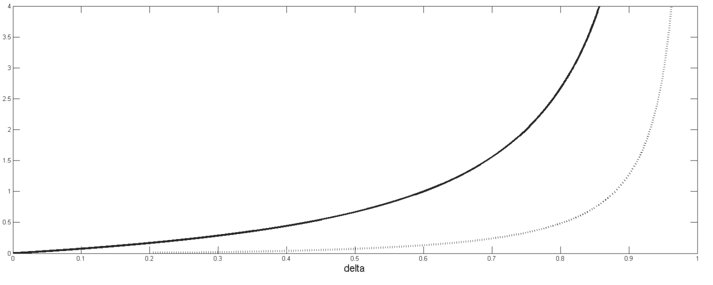


Figure A.1: ε_1 (solid line) and ε_2 (dotted line)

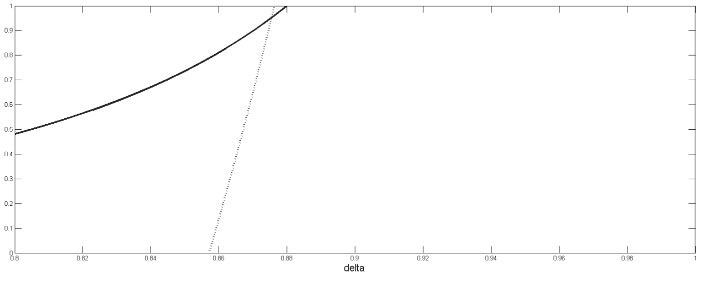


Figure A.2: Region D is below the solid and dotted lines

A.1.5 Region E

Consider the situation that all agents use mixed strategies among all possible proposals. In this situation, we have that $e^0(12) = e^0(13) = e^0(N)$. Then it follows immediately that

$$v = \frac{3-2\delta}{3\delta}\varepsilon, v_3 = \frac{3-2\delta}{3\delta}(1+\varepsilon).$$

For the mixed strategies, the following equations hold

$$\begin{aligned}\sigma_1(12) + \sigma_1(13) + \sigma_1(N) &= 1,\\ \sigma_2(12) + \sigma_2(23) + \sigma_2(N) &= 1,\\ \sigma_3(13) + \sigma_3(23) + \sigma_3(N) &= 1.\end{aligned}$$

For the equilibrium payoff functions, we have

$$\begin{aligned}v &= \tfrac{1}{3}\left(1+\varepsilon-2\delta v-\delta v_3\right) + \tfrac{1}{3}\delta v[1+1-\sigma_2(23)+1-\sigma_3(23)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_2(23)+\sigma_3(23)],\\ v &= \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v[1+1-\sigma_1(13)+1-\sigma_3(13)] + \tfrac{1}{3}\delta\frac{\varepsilon}{3}[\sigma_1(13)+\sigma_3(13)],\\ v_3 &= \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3) + \tfrac{1}{3}\delta v_3[1+1-\sigma_1(12)+1-\sigma_2(12)] + \tfrac{1}{3}\delta\frac{1+\varepsilon}{3}[\sigma_1(12)+\sigma_2(12)].\end{aligned}$$

Since agents 1 and 2 are symmetric, we have $\sigma_1(13) + \sigma_3(13) = \sigma_2(23) + \sigma_3(23)$. Furthermore, we assume

$$\sigma_1(13) + \sigma_3(13) = \sigma_2(23) + \sigma_3(23) = x, \;\; \sigma_1(12) + \sigma_2(12) = y.$$

It is easy to verify that $x \in [0, 1.5]$ and $y \in [0, 2]$.[6] Then we can write down the payoff functions in terms of x and y,

$$\begin{aligned}v &= \tfrac{1}{3}\left(1+\varepsilon-\delta(v_3-v)\right) + \tfrac{1}{9}\delta\left(\varepsilon-3v\right)x,\\ v_3 &= \tfrac{1}{3}\left(1+\varepsilon+2\delta(v_3-v)\right) + \tfrac{1}{9}\delta\left(1+\varepsilon-3v_3\right)y.\end{aligned}$$

Solving the above two equations yields

$$\begin{aligned}x &= \frac{2\delta^2-9\varepsilon\left(1-\delta\right)}{3\delta\varepsilon(1-\delta)},\\ y &= \frac{2\delta\left(3-2\delta\right)-9\left(1-\delta\right)\left(1+\varepsilon\right)}{3\delta\left(1-\delta\right)\left(1+\varepsilon\right)}.\end{aligned}$$

Imposing $x \in [0, 1.5]$ yields

$$\tfrac{4\delta^2}{9(2+\delta)(1-\delta)} \le \varepsilon \le \tfrac{2\delta^2}{9(1-\delta)}; \tag{A.33}$$

imposing $y \in [0, 2]$ yields

$$\tfrac{2\delta(3-2\delta)}{3(3+2\delta)(1-\delta)} - 1 \le \varepsilon \le \tfrac{2\delta(3-2\delta)}{9(1-\delta)} - 1. \tag{A.34}$$

[6] To see this, note that $\sigma_1(13) + \sigma_2(23) + \sigma_3(13) + \sigma_3(23) \le 3$. Hence, $2x \le 3$ and we get $x \le 1.5$.

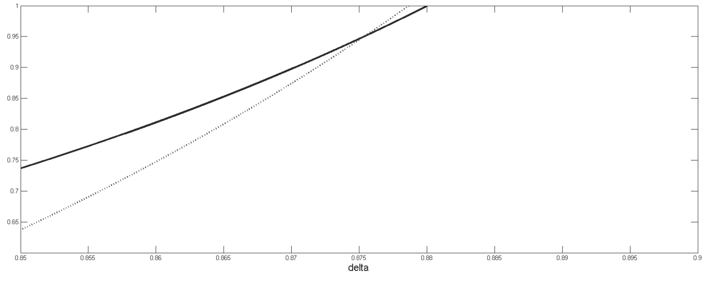


Figure A.3: Region E: below $\frac{(3-\delta)(4\delta-3)}{9(1-\delta)}$ (dotted line) and above ε_2 (solid line).

Furthermore, notice that $2x + y \leq 3$ since $\sigma_1(12) + \sigma_1(13) + \sigma_2(12) + \sigma_2(23) + \sigma_3(13) + \sigma_3(23) \leq 3$. We obtain

$$2\frac{2\delta^2 - 9\varepsilon(1-\delta)}{3\delta\varepsilon(1-\delta)} + \frac{2\delta(3-2\delta) - 9(1-\delta)(1+\varepsilon)}{3\delta(1-\delta)(1+\varepsilon)} \leq 3.$$

By further simplification, we obtain,

$$\varepsilon \geq \frac{9 - 8\delta - 3\delta^2 - \sqrt{81 - 144\delta + 58\delta^2 + 16\delta^3 - 7\delta^4}}{6(\delta+3)(\delta-1)} = \varepsilon_2. \tag{A.35}$$

Combining constraints (A.33), (A.34) and (A.35), we get Figure A.3. And this two boundaries correspond exactly with Region C (i.e., the constraint $\frac{2\delta(3-2\delta)}{9(1-\delta)} - 1$) and Region D (i.e., the constraint $\frac{9-8\delta-3\delta^2-\sqrt{81-144\delta+58\delta^2+16\delta^3-7\delta^4}}{6(-3+2\delta+\delta^2)}$). Considering when $y = 0$, indeed it corresponds to Region C. Considering when $2x + y = 3$, namely $\sigma_1(N) = \sigma_2(N) = \sigma_3(N) = 0$, indeed it corresponds to Region D.

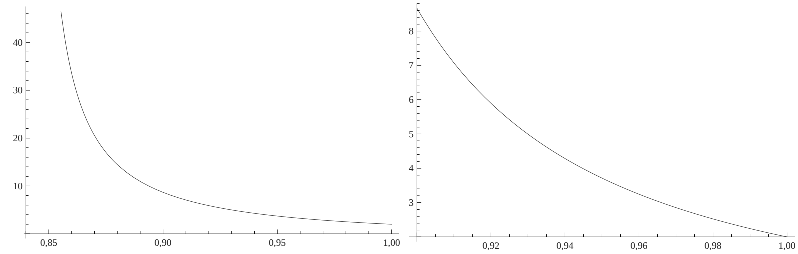
A.2 Impossible cases

A.2.1 Inequality (A.8)

When $e^0(12) > e^0(13) = e^0(N)$, then agents 1 and 2 will propose $\{1,2\}$ and agent 3 will mix $\{1,3\}$, $\{2,3\}$ and the grand coalition. Since agents 1 and 2 are symmetric in this setting, we assume that $\sigma_3(13) = \sigma_3(23) = x \in [0,1]$ and it follows that $\sigma_3(N) = 1 - 2x$. It follows immediately from the condition $e^0(12) > e^0(13) = e^0(N)$ that $v = \frac{3-2\delta}{3\delta}\varepsilon$ and $\delta v_3 > \frac{1}{3}(1+\varepsilon)(3-2\delta)$. For the equilibrium payoff functions, we have

$$v = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)\delta}{3} - 2\delta v\right] + \tfrac{1}{3}\delta v(3-x) + \tfrac{1}{3}\delta\frac{\varepsilon}{3}x,$$
$$v_3 = \tfrac{1}{3}\left(1 + \varepsilon - 2\delta v - \delta v_3\right) + \tfrac{1}{3}\delta v_3 1 + \tfrac{1}{3}\delta 2\frac{1+\varepsilon}{3}.$$

Figure A.4: Infimum of x



Infimum for $\delta \in (0.842329, 1]$ Infimum for $\delta \in [0.9, 1]$

From the payoff function of agent 3, we obtain $v_3 = \frac{3-3\varepsilon+2\delta+6\delta\varepsilon}{9}$. From the payoff function of agent 1 together with $v = \frac{3-2\delta}{3\delta}\varepsilon$, we obtain $x = \frac{2\delta^2-9\varepsilon+9\delta\varepsilon}{3\delta\varepsilon(1-\delta)}$. It can be easily verified that x is decreasing in ε. Still we need to consider the condition $\delta v_3 > \frac{1}{3}(1+\varepsilon)(3-2\delta)$, and we obtain $\varepsilon < \frac{2\delta^2+9\delta-9}{9-3\delta-6\delta^2}$. For $\varepsilon \in [0,1]$, we get the restriction that $2\delta^2+9\delta-9 > 0$, so $\delta \in (0.842329, 1]$.

Next we will show that for $\delta \in (0.842329, 1]$, x is bounded away from $[0,1]$. Since x is decreasing in ε and $\varepsilon < \frac{2\delta^2+9\delta-9}{9-3\delta-6\delta^2}$, then for every $\delta \in (0.842329, 1]$, we get the infimum for x by setting $\varepsilon = \frac{2\delta^2+9\delta-9}{9-3\delta-6\delta^2}$. In this situation,

$$x_{infimum} = \frac{2\delta^2(9-3\delta-6\delta^2)+9(\delta-1)(2\delta^2+9\delta-9)}{3\delta(1-\delta)(2\delta^2+9\delta-9)}.$$

Indeed we can see from Figure A.4 that the infimum for x is bounded away from [0,1]. Hence, we cannot find an equilibrium in this situation.

A.2.2 Inequality (A.9)

When $e^0(12) > e^0(13) > e^0(N)$, agents 1 and 2 will propose $\{1,2\}$ and agent 3 will mix $\{1,3\}$ and $\{2,3\}$ with equal probabilities $\frac{1}{2}$. From the condition $e^0(12) > e^0(13) > e^0(N)$, we have

$$\delta v > \tfrac{1}{3}(3-2\delta)\varepsilon,\ \delta(v_3 - v) > 1 - \tfrac{2}{3}\delta.$$

For the equilibrium payoff functions, we have

$$v = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)\delta}{3} - 2\delta v\right] + \tfrac{1}{3}\delta v(1+1+\tfrac{1}{2}) + \frac{1}{3}\delta\tfrac{1}{2}\frac{\varepsilon}{3},$$
$$v_3 = \tfrac{1}{3}\left[\frac{(3+2\varepsilon)\delta}{3} + (1-\delta) - \delta v - \delta v_3\right] + \tfrac{1}{3}\delta v_3 1 + \tfrac{1}{3}\delta 2\frac{1+\varepsilon}{3}.$$

Solving the above equations, we obtain

$$v = \frac{4\delta+5\delta\varepsilon}{3(6-\delta)},\ v_3 = \tfrac{1}{3}\left(\frac{2\varepsilon\delta}{3} + 1 - \delta\frac{4\delta+5\delta\varepsilon}{3(6-\delta)}\right) + \frac{2\delta(1+\varepsilon)}{9}.$$

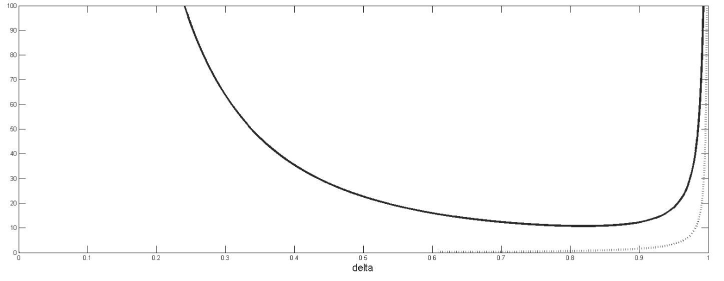


Figure A.5: $\frac{3(\delta^3+3\delta^2-21\delta+27)}{9\delta^2(1-\delta)}$ (solid line) and $\frac{4\delta^2}{3(\delta+6)(1-\delta)}$ (dotted line)

Still we need to check for the two conditions. From $\delta v > \frac{1}{3}(3-2\delta)\varepsilon$, we obtain

$$\varepsilon < \frac{4\delta^2}{3(\delta+6)(1-\delta)}. \tag{A.36}$$

From $\delta(v_3 - v) > 1 - \frac{2}{3}\delta$, we obtain

$$\varepsilon > \frac{3(\delta^3+3\delta^2-21\delta+27)}{9\delta^2(1-\delta)}. \tag{A.37}$$

For $\delta \in [0,1]$, we plot the two bounds (i.e., (A.36) and (A.37)) in Figure A.5. It is clear that we cannot find such ε that is above the solid line and below the dotted line. Hence, we cannot find an equilibrium in this situation as well.

A.2.3 Inequality (A.15)

When $e^0(12) = e^0(N) > e^0(13)$, then agents 1 and 2 will mix over $\{1,2\}$ and the grand coalition and agent 3 will propose the grand coalition. It follows immediately from the condition $e^0(12) = e^0(N) > e^0(13)$ that

$$v_3 = (1+\varepsilon)(\frac{1}{\delta} - \frac{2}{3}),\ \delta v < \frac{1}{3}\varepsilon(3-2\delta).$$

For the equilibrium payoff functions, we have

$$v = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3)+\delta v,$$
$$v_3 = \tfrac{1}{3}(1+\varepsilon-2\delta v-\delta v_3)+\tfrac{1}{3}\left[1+\sigma_1(N)+\sigma_2(N)\right]+\tfrac{1}{3}\delta\frac{1+\varepsilon}{3}\left[\sigma_1(12)+\sigma_2(12)\right].$$

From the payoff function of agent 1, we get that $v = \frac{2(1+\varepsilon)\delta}{3(3-\delta)}$. Let $\sigma_1(N)+\sigma_2(N) = x \in [0,2]$, then $\sigma_1(12)+\sigma_2(12) = 2-x$. From the payoff function of agent 3, we have

$$v_3 = \frac{(1+\varepsilon)(3-\frac{4\delta^2}{3-\delta}+\delta(2-x))}{9-3\delta x}.$$

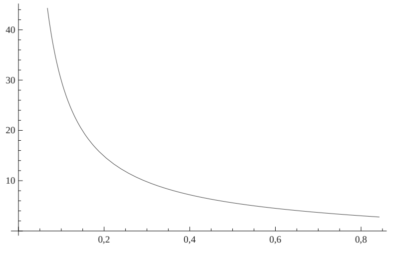


Figure A.6: $x = \frac{(3-2\delta)(\delta+3)}{\delta(3-\delta)}$ for $\delta \in (0, 0.842329)$

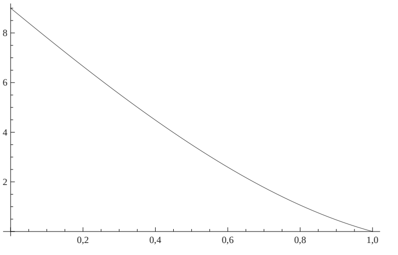


Figure A.7: $2\delta^3 + \delta^2 - 12\delta + 9$ for $\delta \in [0,1]$

Equating the two expressions we get for v_3, i.e.,

$$\frac{(1+\varepsilon)(3 - \frac{4\delta^2}{3-\delta} + \delta(2-x))}{9 - 3\delta x} = (1+\varepsilon)(\frac{1}{\delta} - \tfrac{2}{3}),$$

we obtain $x = \frac{(3-2\delta)(\delta+3)}{\delta(3-\delta)}$. Still we need to consider the condition $\delta v < \frac{1}{3}\varepsilon(3-2\delta)$ and we get $\varepsilon > \frac{2\delta^2}{9(1-\delta)}$. For $\varepsilon \in [0,1]$ has a solution, we need to impose the restriction that $\frac{2\delta^2}{9(1-\delta)} < 1$ and we get $\delta \in (0, 0.842329)$. However, for $\delta \in (0, 0.842329)$, x is always above 2 and it is depicted in Figure A.6. Hence, we cannot find an equilibrium in this situation.

A.2.4 **Inequality** (A.16)

When $e^0(12) > e^0(N) > e^0(13)$, then agents 1 and 2 will propose {1,2} and agent 3 will propose the grand coalition. From the condition $e^0(12) > e^0(N) > e^0(13)$, we get

$$\delta v_3 > \tfrac{1}{3}(1+\varepsilon)(3-2\delta),\ \delta v < \frac{\varepsilon}{3}(3-2\delta).$$

For the equilibrium payoff functions, we have

$$v = \tfrac{1}{3}\left[\frac{2(1+\varepsilon)\delta}{3} - 2\delta v\right] + \tfrac{1}{3}\delta v 3,$$
$$v_3 = \tfrac{1}{3}\left[1+\varepsilon - 2\delta v - \delta v_3\right] + \tfrac{1}{3}\delta v_3 1 + \tfrac{1}{3}\delta\frac{1+\varepsilon}{3}2.$$

Solving the above two equations, we obtain

$$v = \frac{2(1+\varepsilon)\delta}{3(3-\delta)},\ v_3 = \tfrac{1}{3}\left[(1+\varepsilon)\left(1 - \frac{4\delta^2}{3(3-\delta)}\right)\right] + \frac{2(1+\varepsilon)\delta}{9}.$$

Still we need to check for the conditions. From $\delta v_3 > \frac{1}{3}(1+\varepsilon)(3-2\delta)$, we obtain $2\delta^3 + \delta^2 - 12\delta + 9 < 0$. In Figure A.7, we plot the function in the range of $\delta \in [0,1]$ and obviously in that range it is impossible for $2\delta^3 + \delta^2 - 12\delta + 9 < 0$ to hold.