Chapter 2

Some concepts in Game Theory

2.1 Asymmetric Nash Bargaining Solution

2.1.1 Formal description of the two person bargaining problem

A two person bargaining problem in utility representation consists of a disagreement point $d = (d_1, d_2) \in U$ and a feasibility set $U \in \mathbb{R}^2$. The disagreement point d is the payoff the players can expect to get if negotiations break down. A feasibility set U is a closed, convex and compact subset of $\mathbb{R}^2$ and the elements of the feasibility set are interpreted as agreements. An agreement can take many forms, e.g., it could be an employment contract between an employer and an employee, or a trade agreement between two countries. In our water resource allocation problem, an agreement could be the water resources allocated to each country and the monetary compensation scheme among all countries within a trans-boundary river basin. The bargaining problem is nontrivial if there is at least one agreement in U that is better for both parties than the disagreement, i.e., there exists some $v \in U$ such that $v > d$ (i.e., $v_i > d_i$ for $i = 1, 2$). Formally, we denote a bargaining problem as (U, d) and the set of all possible bargaining problems by $\mathcal{B}$. A bargaining solution is a function $f : \mathcal{B} \to \mathbb{R}^2$, s.t. $\forall (U, d) \in \mathcal{B}, f(U, d) \in U$.

2.1.2 Nash bargaining solution

Nash (1950) proposes a bargaining solution that satisfies the following four axioms: invariance to affine transformation, Pareto efficiency, independence of irrelevant alternatives and symmetry. These axioms are defined as follows,

- Invariance to affine transformations: given a bargaining problem (U, d), consider the bargaining problem (U', d'), where for some $\alpha_i > 0, i = 1, 2$ and $\beta_i, i = 1, 2$:

$$\begin{aligned} U' &= \{(\alpha_1 v_1 + \beta_1, \alpha_2 v_2 + \beta_2) | (v_1, v_2) \in U\}, \\ d' &= (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2). \end{aligned}$$

  Then, $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i, i = 1, 2, \forall (U, d) \in \mathcal{B}$.
- Pareto efficiency: a bargaining solution f is Pareto efficient if for $(U, d) \in \mathcal{B}$ there does not exist a $(v_1, v_2) \in U$ such that $v \geq f(U, d)$ and $v_i > f_i(U, d)$ for some i.

- Independence of irrelevant alternatives: let (U, d) and (U', d) be two bargaining problems such that $U' \subset U$. If $f(U, d) \in U'$, then $f(U', d) = f(U, d)$.
- Symmetry: for all $(U, d) \in \mathcal{B}$, if $d_1 = d_2$ and $(v_2, v_1) \in U$ if and only if $(v_1, v_2) \in U$, then $f_1(U, d) = f_2(U, d)$.

The above four axioms state the following: a transformation of the utility function that maintains the same ordering over preferences (such as a linear transformation) should not alter the outcome of the bargaining process; an inefficient outcome is not allowed, since it always leaves some space for renegotiation; if we take irrelevant alternatives from the original bargaining problem, the outcome of the bargaining process remains the same; if the players are indistinguishable, the agreement should not discriminate between them as well.

Nash (1950) proves that there is a unique function f that satisfies the above mentioned four axioms on the class $\mathcal{B}$ of all bargaining problems. We state the following result without a formal proof.

Proposition 2.1. *Nash bargaining solution is the unique bargaining solution that satisfies the above mentioned four axioms, i.e., invariance to affine transformations, Pareto efficiency, independence of irrelevant alternatives and symmetry.*

In the following, we denote the unique function that satisfies the four axioms by f^N. Next, we propose a way to find $f^N(U, d)$ for any given $(U, d) \in \mathcal{B}$.

Definition 2.1. *For any given $(U, d) \in \mathcal{B}$, $f^N(U, d)$ yields the pair of payoffs $v^* = (v_1^*, v_2^*)$, where v^* solves the optimization problem*

$$\begin{aligned} \max_{v_1, v_2} \quad & (v_1 - d_1)(v_2 - d_2) \\ s.t. \quad & (v_1, v_2) \in U, \ \text{and} \ (v_1, v_2) \geq (d_1, d_2). \end{aligned} \tag{2.1}$$

Since the set U is compact and the objective function of problem (2.1) is continuous, there exists an optimal solution for problem (2.1). Moreover, the objective function of problem (2.1) is strictly quasi-concave and U is a convex set, therefore problem (2.1) has a unique optimal solution.

2.1.3 Asymmetric Nash bargaining solution

The two person bargaining problem can be generalized to n person. The set of players is denoted as $N = \{1, 2, \cdots, n\}$. Let $U \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$. The class of bargaining problems for n person is denoted as $\mathcal{B}^n = \{(U, d) | U \in \mathbb{R}^n, d \in U\}$. For given bargaining weights $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, with $\sum_{i=1}^n \alpha_i = 1$, the asymmetric bargaining solution is the function $f^\alpha : \mathcal{B}^n \to \mathbb{R}^n$ assigning outcome $f^\alpha(U, d) \in U$, s.t. $f^\alpha(U, d) \geq d$, $\forall (U, d) \in \mathcal{B}^n$. Similarly, we construct the following maximization problem to find $f^\alpha(U, d)$ for any $(U, d) \in \mathcal{B}$,

$$\begin{aligned} \max_{v_1, v_2, \cdots, v_n} \quad & \Pi_{i=1}^n (v_i - d_i)^{\alpha_i} \\ \text{s.t.} \quad & (v_1, v_2, \cdots, v_n) \in U, \text{ and } (v_1, v_2, \cdots, v_n) \geq (d_1, d_2, \cdots, d_n). \end{aligned} \tag{2.2}$$

In the two person bargaining problem, f^N is the solution for $\alpha_1 = \alpha_2 = \frac{1}{2}$. Hence, in the original Nash bargaining solution, the bargaining weights are taken to be equal for both players.

The Nash bargaining solution has been widely applied in many economic contexts, e.g, the wage bargaining between employers and employees, see, Binmore et al. (1986). In economic applications, bargaining weights are often related to GDP, population sizes, political factors, military powers, etc and taken to be asymmetric. This reflects that countries with, say, a larger GDP have much more at stake internationally as well as more financial means to maintain a large and well-trained corpse of diplomats and negotiators, or a large army to conduct geopolitics. The strategic bargaining literature underpins bargaining weights as either the probability of setting the agenda in random proposer bargaining or the differences in individual time-preferences, see e.g., Herrero (1989), Miyakawa (2006), Laruelle and Valenciano (2008) and Herings and Predtetchinski (2010). In our trans-boundary river sharing problems in Chapter 3, we apply a general version of the Nash bargaining solution and also the bargaining weights are taken to be asymmetric due to the reason that countries within a river basin differ in their population sizes, GDP, political factors and military strengths, etc.

2.2 Coalitional Bargaining Game

In this section, we introduce the coalitional bargaining game in its most general form with a partition function form. In Chapter 4, we will use this coalitional bargaining game to investigate a specific trans-boundary river sharing problem. Note that a partition function is a generalization of the characteristic function of a coalition formation game. The characteristics function, firstly introduced by von Neumann and Morgenstern (1944), assigns a value to each coalition independent of the action taken by other players. While the partition function assigns a value to each coalition dependent on the actions taken by other coalitions (or players).

2.2.1 The Coalitional Bargaining Game

In this section, we introduce the general coalitional bargaining game (hereafter, CBG) of Gomes (2005). The set of players is denoted as $N = \{1, 2, ..., n\}$. A coalition structure π is a partition of N describing which coalitions have formed already. The initial coalition structure is denoted by π^0 and it is the finest partition of N, i.e., $\pi^0 = \{\{1\}, \{2\}, ..., \{n\}\}$. The CBG is a dynamic game where, at every round, a player (being a coalition in π) is randomly chosen to propose an offer to a set of other players (coalitions) in the coalition structure, who can either accept or decline the offer. For ease of notation, we assume invariant probabilities in selecting who proposes throughout all rounds and all coalition structures. Formally, there is a probability distribution $p = (p_i)_{i \in N}$ with $p_i > 0$ for all i, and at the beginning of a bargaining round τ with coalition structure π formed in the previous rounds, the probability of a coalition $C \in \pi$ is selected to be the proposer with probability $p_C = \sum_{k \in C} p_k$.[1]

The rules of the CBG are as follows. Let π be the coalition structure at the start of some bargaining round τ and $S \subseteq \pi$ be a collection of coalitions in π. Then the coalitions in S can bargain and form a new coalition $C' = \cup_{B \in S} B$, changing the coalition structure to $\pi' = \{C'\} \cup (\pi \setminus S)$, where C' is the set of all players in N belonging to one of the coalitions in S. To be more precise, at the start of a round a coalition C in the coalition structure π is randomly chosen with probability p_C to be the proposer. This coalition then proposes the pair (S, t), where $S \subseteq \pi$ with $C \in S$, is a collection of coalitions that are proposed to merge into C', and the vector $t = (t_B)_{B \in S \setminus \{C\}}$ consists of the proposed monetary offers $t_B \in R$ made by C to each coalition $B \in S \setminus \{C\}$. The coalitions in the collection S that are proposed to, i.e., the coalitions in $S \setminus \{C\}$, respond sequentially in a fixed order (the order of response turns out to be irrelevant), either accepting or rejecting the offer. If these coalitions are unanimous in accepting the offer, then the new coalition structure $\pi' = \{C'\} \cup (\pi \setminus S)$ immediately forms in round τ and the proposing coalition C divides the continuation payoff of coalition C' in π' by paying every joining coalition $B \in S$ the proposed lump-sum transfer t_B and keeping the remainder.[2] If the responders are not unanimous in accepting the offer, then the next round's coalition structure equals this round's coalition structure, i.e., the coalition structure does not change, and there are no payments. We then write $\pi' = \pi$. After accepting or rejecting the proposal the game ends when $\pi' = \{N\}$, (which can only happen when the proposal (π, t) was accepted). Otherwise, the game proceeds into the next round where the stage game is played once again and a coalition from π' is chosen randomly to be the proposer. Notice that according to the rules that once a coalition has formed, it never breaks up[3] and the renegotiation possibilities only allow for further merging of coalitions in the coalition structure. Furthermore it is assumed that all players have perfect information about the past play in the game when they make decisions. In order to avoid confusion, we will refer to payoffs per round as disagreement values.

- disagreement value: $d_C(\pi)$ is the disagreement payoff for coalition C in π, i.e., it is the value that coalition C obtains per round when π is the coalition structure.

Notice that even for $\pi = \{N\}$ we call $d_N(\{N\})$ the disagreement value for the grand coalition. Infinite streams of disagreement values are discounted by the common discount factor $\delta \in [0, 1)$. Formally, we will work with normalized discounted payoffs $(1 - \delta) \sum_{\tau=0}^{\infty} \delta^\tau x^\tau$, where x^τ denotes the payoff in round τ. The merit of this normalization is that if we let δ go to 1, then the normalized sum remains bounded.

Finally, following Gomes (2005), we will investigate mixed strategies and we assume that all players (coalitions) in this game have von Neumann-Morgenstern preferences.

[1]Our general results also hold if we would allow for general probabilities $p_C(\pi)$ that depend on the coalition structure. Okada (1996, 2000, 2011) also assumes invariant probabilities.

[2]Gomes (2005) assumes that coalition C becomes the sole owner of coalition C' after buying off all owners $B \in S \setminus \{C\}$. Alternatively, but more elaborate, we argue to consider the members of C' as shareholders whose share of current and future disagreement values is worth the offers in t_B.

[3]Readers interested in a setting where coalitions may break up are referred to Gomes and Jehiel (2005).

2.2.2 Markov Perfect Equilibria

In this section, we follow Gomes (2005) and study mixed Markov perfect equilibrium (hereafter, MPE) of the general CBG. As is shown in this reference, mixed MPEs are necessary for the existence of MPEs and therefore unavoidable in any analysis. We first explain how to characterize mixed MPEs and then derive a necessary and sufficient condition for the immediate emergence of the grand coalition in an MPE. We do so in separate subsections.

2.2.2.1 Strategies and Equilibria

Markov strategies assume that the τ-th round strategy of a player depends only on the 'state' of the game at the start of round τ. In the CBG, the state in round τ is the coalition structure at the start of round τ. Thus, Markov strategies only depend on the coalition structure π. At the stage game of round τ, the coalitions in the coalition structure make decisions: one coalition makes a proposal and others who are proposed to respond. Therefore, following Gomes (2005), we assume that in every stage game the strategies of all players that belong to a same coalition C coincide, i.e., players in a coalition act together as one decision maker. Under this assumption we define for every coalition structure π a stage-game strategy for every coalition $C \in \pi$, with the interpretation that the stage-game strategy of $C \in \pi$ is the stage-game strategy of every player $i \in C$, avoiding in this way duplication of strategies. Given a coalition structure π, the pure stage-game strategy of coalition $C \in \pi$ is composed of a proposal when selected as the proposer and a response when being proposed to. In a (behavioral) mixed stage-game strategy of coalition $C \in \pi$, this coalition possibly randomizes over the set of actions available when taking the decision. Thus, coalition C randomizes over the set of all proposals (S, t) when being the proposer and also randomizes over accept and reject when being proposed to.

Mixed Markov strategies induce present values of infinite streams of expected disagreement values, simply called values, for every coalition that depend on the coalition structure π. In addition, we refer to these values as the equilibrium values in a MPE. So, we have the following notion of value.

- $v_C(\pi)$ is the continuation (MPE) value for coalition C in coalition structure π at the beginning of round τ prior to the randomization that determines who becomes the proposing player in this round.

These values together with the stage-game Markov strategies have to be determined by the equilibrium analysis. Notice that the value for the grand coalition is trivial and equal to the infinite stream of disagreement values under coalition structure $\pi = \{N\}$, i.e.,

$$v_N(\{N\}) = (1-\delta)\sum_{\tau=0}^{\infty} \delta^\tau d_N(\{N\}) = d_N(\{N\}).$$

So when the grand coalition is formed, the value of N is equal to the disagreement value.

As shown in Gomes (2005) for any coalition structure, the response of any coalition in any MPE stage-game strategy is always a pure strategy in which the responder always accepts when indifferent. Formally, when a coalition $C \in \pi$ has to respond to a proposal (S,t) made by another coalition $C' \in \pi$, i.e., $C \in S \subset \pi$ and t is a vector of offers including an offer t_C of C' to C, then C accepts if and only if $t_C \geq t_C(\pi)$, where $t_C(\pi)$ is the threshold value of C in π defined as

$$t_C(\pi) = \delta v_C(\pi) + (1-\delta) d_C(\pi). \tag{2.3}$$

When C rejects the offer, it gets discounted continuation value plus the normalized disagreement value in this round. For future reference, we write $t^S(\pi) = (t_B(\pi))_{B \in S \setminus \{C\}}$.

Now we consider the pure stage-game strategy of a coalition $C \in \pi$ when C is chosen to be the proposer. When $C \in \pi$ proposes (S,t), the responding coalitions in S will all accept the proposal if and only if the offer to any $B \in S \setminus \{C\}$ is at least equal to its threshold value $t_B(\pi)$. It follows that C extracts the highest surplus from proposing S when offering exactly these threshold values to every $B \in S \setminus \{C\}$. When C proposes $(S, t^S(\pi))$ and all responding coalitions in S accept, then coalition $C' = \cup_{B \in S} B$ immediately forms, $\pi' = \{C'\} \cup (\pi \setminus S)$ immediately becomes the new coalition structure and the value to coalition C is equal to

$$v_C((S, t^S(\pi))|\pi) = \delta v_{C'}(\pi') + (1-\delta) d_{C'}(\pi') - \sum_{B \in S \setminus \{C\}} t_B(\pi). \tag{2.4}$$

This value is equal to the discounted continuation value $v_{C'}(\pi')$ under the new coalition structure π' plus the normalized disagreement value under π' minus the sum of the equilibrium offers to the joining coalitions in S. The maximal attainable surplus that coalition C can obtain from proposing any $(S, t^S(\pi))$ is defined as

$$v_C^*(\pi) = \max_{S \subseteq \pi, S \ni C} v_C((S, t^S(\pi))|\pi). \tag{2.5}$$

So, when $C \in \pi$ is chosen to be the proposer and it plays a pure MPE stage-game strategy (S,t), then the proposed S of this MPE strategy must be a maximizer of (2.5) and the associated transfers $t^S(\pi)$ offer exactly the threshold value $t_B(\pi)$ to any $B \in S \setminus \{C\}$. Notice that we allow that coalition C proposes $S = \{C\}$, i.e., it proposes to stay alone, guaranteeing itself $\delta v_C(\pi) + (1-\delta) d_C(\pi)$. Staying alone is the unique best response when $v_C((S', t^{S'}(\pi))|\pi) < \delta v_C(\pi) + (1-\delta) d_C(\pi)$ for every S' containing some coalition $B \in \pi \setminus \{C\}$.

The existence result for MPE in Gomes (2005) is for mixed MPE in which a proposing coalition C is allowed to randomize over several coalitions S that are maximizers of (2.5) with the understanding that the MPE proposal is $\big(S, t^S(\pi)\big)$. Let $\sigma_C(S|\pi) \geq 0$ be the probability that C proposes $(S, t^S(\pi))$ with $\sum_{S \subset \pi, S \ni C} \sigma_C(S|\pi) = 1$. Then a mixed Markov strategy for the proposal of C in a stage game with coalition structure π is optimal if and only if

$$\sigma_C(S|\pi) = 0 \text{ if } v_C((S, t^S(\pi))|\pi) < v_C^*(\pi) \text{ for all } S \subseteq \pi, S \ni C. \tag{2.6}$$

We end this section with the value of $C \in \pi$ in a mixed strategy MPE. When C is chosen to be the proposer it realizes payoff $v_C^*(\pi) = \max_{S \subseteq \pi, S \ni C} v_C((S, t^S(\pi))|\pi)$ from playing an optimal mixed Markov strategy. This happens with probability p_C. When some other coalition $B \in \pi$ is the proposer and proposes S including C, then C gets it threshold value $t_C(\pi)$. This happens with probability $\sum_{B\in\pi\setminus\{C\}}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C}}\sigma_B(S|\pi)\right]$. Finally, when B proposes a collection S not including C, then the expected payoff of C is given by $t_C(\pi')$, where π' is the coalition structure resulting from the proposal of B. This happens with probability $\sum_{B\in\pi\setminus\{C\}}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C\notin S}}\sigma_B(S|\pi)\right]$. It follows that

$$v_C(\pi)=p_C v_C^*(\pi)+t_C(\pi)\sum_{B\in\pi\setminus\{C\}}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C}}\sigma_B(S|\pi)\right]+t_C(\pi')\sum_{B\in\pi\setminus\{C\}}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C\notin S}}\sigma_B(S|\pi)\right].$$

Rewriting the above formula, we obtain,

$$\begin{aligned} v_C(\pi) &= p_C(v_C^*(\pi)-t_C(\pi)) \\ &+t_C(\pi)\sum_{B\in\pi}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C}}\sigma_B(S|\pi)\right]+t_C(\pi')\sum_{B\in\pi\setminus\{C\}}\left[p_B\sum_{\substack{S\subseteq\pi\\ S\ni B,C\notin S}}\sigma_B(S|\pi)\right]. \end{aligned} \tag{2.7}$$

In the above expression, the value function is composed of two parts: the first part (the first term of the right-hand side) is the maximal net surplus from being the proposer if all coalitions $B\in S$ including proposing coalition C are paid $t_B(\pi)$. This net surplus is the advantage to propose. Furthermore, for every merger of coalitions $B\in S$, the net surplus of proposing the coalition S is the same. In MPE, each coalition in π randomizes over those S that maximize this coalition's net surplus, see Gomes (2005). The second part (the second and third term of the right-hand side) represents coalition C's status quo value. Note that the status quo value depends on other players' mixed strategies as well. For instance, the second term is the status quo value when C is included in the proposal and the third term describes the situation when C is not included in the proposal.

2.2.2.2 Immediate Formation of the Grand Coalition

In this section we first impose a sufficient condition that assures that in every bargaining round at least two coalitions in the coalition structure merge together, i.e., under the assumption every proposing coalition proposes to merge with at least one other coalition. It follows that under this assumption the grand coalition forms in at most $n-1$ rounds, because in a MPE a proposal is always accepted.

Assumption 2.1. *For every coalition structure $\pi\neq\{N\}$ it holds that $\sum_{B\in\pi} d_B(\pi)< d_N(\{N\})$.*

Lemma 2.1 (**Gomes, 2005**). *Under Assumption 2.1 it holds that for every $\pi\neq\{N\}$ and every $C\in\pi$ that $\sigma_C(\{C\}|\pi)=0$.*

Although Gomes (2005) provided a proof, we will give an alternative proof in the appendix of this chapter.

Assumption 2.1 means that only the grand coalition can fully internalize the externalities and therefore it follows that the maximum surplus from being the proposer (i.e., the first term of the right-hand of equation (2.7)) is positive. Therefore in every coalition structure every proposer will propose to merge with at least one other coalition, thus $\sigma_C(\{C\}|\pi) = 0$. Notice that, in any mixed MPE, the proposer may randomize over several coalitions and that he always makes proposals of the form $(S, t^S(\pi))$ that will be accepted. So, the proposer will reach some agreement for sure.

The consequence of this result is that each coalition $C \in \pi$ will make an acceptable proposal for sure and that the state moves from π to $\pi^{'}$, where $S \neq \{C\}$ depends upon the coalition proposed by C. An intuitive explanation is that players can renegotiate the contract and the efficient grand coalition acts as a "sink" of the Markov process to which the system converges. Similar results can be found in the literature (e.g., Chatterjee et al. (1993), Okada (2000)). The fact that the equilibrium path converges to the grand coalition is based on the assumptions that the coalition does not break up and the efficiency of the grand coalition.

We next state several results under the assumption that it is optimal to propose the grand coalition. The first lemma yields the continuation payoff of a coalition $C \in \pi$ when it is optimal for every $B \in \pi$ to propose $S = \pi$, thus to propose the grand coalition $N = \cup_{D\in\pi} D$ with probability 1.

Lemma 2.2 (Gomes, 2005). *For a coalition structure $\pi \neq \{N\}$, let $\sigma_B(\pi|\pi) = 1$ for every $B \in \pi$. Then it holds that*

$$v_C(\pi) = d_C(\pi) + p_C\left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right), \text{ for every } C \in \pi.$$

In Gomes (2005), this result is stated without proof and we provide a proof in the appendix of this chapter.

The lemma states that the value for a coalition $C \in \pi$ when every coalition in π proposes to form the grand coalition is equal to its disagreement payoff $d_C(\pi)$ under coalition structure π plus a fraction p_C of the net surplus $\left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right)$. The probability to propose can be regarded as a measure of bargaining power and be taken as a coalition's bargaining weight in the asymmetric Nash bargaining solution.

In order to state the next result, we need some additional notation. For two coalition structures π and π', we denote $\pi' \leq_p \pi$ if $\pi' \neq \pi$ can be obtained from π by merging coalitions together, i.e., π' is coarser than π.[4] The next theorem states that under Assumption 2.1 it is optimal for every proposer in a coalition structure $\pi \neq \{N\}$ and every coarser coalition structure $\pi' \leq_p \pi$ to propose the grand coalition if and only if the discount factor is at most equal to some upper bound. Let $S \subset \pi$ be a proper collection of subsets of π, thus $\cup_{B\in S} B \neq N$, and suppose that a member of S proposes to form $\cup_{B\in S} B$. Let π_S be the new coalition structure, thus $\pi_S = \{S^{\cup}\} \cup (\pi \setminus S)$, see footnote 4 for notation $S^{\cup}$.

[4]Formally, we denote $\pi' \leq_p \pi$ if $|\pi'| \leq |\pi|$ and for every $B \in \pi'$ there exist $S \subseteq \pi$ such that $B = S^{\cup}$, where $S^{\cup}$ denotes $\cup_{B\in S} B$. Note that when $C \in S$ is the proposer, we denoted above $S^{\cup}$ by C' and π_S by $\pi' = \{C' \cup (\pi \setminus S)\}$. So $\pi' \leq_p \pi$ and $\pi \leq_p \pi'$ if and only if $\pi = \pi'$. Moreover, any $\pi' \leq_p \pi$ such that $\pi' \neq \pi$ is obtained from π after merging coalitions together. Notice that $\pi' = \{N\}$ if and only if $|\pi'| = 1$.

Now, define

$$\delta_\pi(S) = \frac{d_N(\{N\}) - d_{S^\cup}(\pi_S) - \sum_{B\in\pi\setminus S} d_B(\pi)}{\left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right)\sum_{B\in\pi\setminus S} p_B + p_{S^\cup}\left(d_N(\{N\}) - \sum_{B\in\pi_S} d_B(\pi_S)\right)} \quad (2.8)$$

and

$$\delta(\pi) = \min_{S^\cup \neq N} \ \delta_\pi(S).$$

Finally we define

$$\overline{\delta}(\pi) = \min_{\{\ \pi'\leq_p\pi\ |\ \pi'\neq\{N\}\ \}} \ \delta(\pi').$$

We show the following result.

Theorem 2.1. *Let $\pi \neq \{N\}$. Under Assumption 2.1, for every proposer in the subgame at state π and every proposer in every subgame at state $\pi' \leq_p \pi$ it is optimal to propose $S = \pi$ if and only if $\delta \leq \overline{\delta}(\pi)$.*

The MPE strategies underlying this result are pure strategies. Intuitively, if δ is large enough, delay of forming the grand coalition might occur because then the efficiency loss due to the delay of the grand coalition formation is relatively small. In such a case proposers may have an incentive to search for a better bargaining position, e.g., propose a partial coalition instead of the grand coalition. When $\delta \leq \overline{\delta}(\pi)$, then the grand coalition will be proposed immediately in π and also in any coalition structure $\pi' \leq_p \pi$, so in any coalition structure $\pi' \leq_p \pi$ that might evolve in the sequence of bargaining rounds that will occur when a coalition $C \in \pi$ deviates from the equilibrium path and the grand coalition does not form immediately from π. Notice that, by definition, $\overline{\delta}(\pi) \leq \delta(\pi)$ and for $\delta > \overline{\delta}(\pi)$ we distinguish $\overline{\delta}(\pi) = \delta(\pi)$ from $\overline{\delta}(\pi) < \delta(\pi)$. First, in case $\overline{\delta}(\pi) = \delta(\pi)$, i.e., the minimum over $\pi' \leq_p \pi$ is attained at the current coalition structure π, then proposing the grand coalition is clearly not optimal in coalition structure π because of $\delta > \overline{\delta}(\pi) = \delta(\pi)$. Second, if $\overline{\delta}(\pi) < \delta(\pi)$, i.e., the minimum is attained for some coarser $\pi' \neq \pi$, then for $\overline{\delta}(\pi) < \delta < \delta(\pi)$, there is some $\pi' \leq_p \pi$, $\pi' \neq \pi$, at which for at least one coalition in π' it is not optimal to propose the grand coalition. As a result it might be that, on the equilibrium path, for some coalition in π it is better to make a proposal leading to π' (in one or more bargaining rounds) instead of moving to the grand coalition immediately. However, in this situation it might also be that, on the equilibrium path, for every coalition in π it is still optimal to move to N immediately, only when a coalition deviates and the off-the-equilibrium-path coalition structure π' is reached it is not optimal anymore to form N, which in equilibrium cannot occur.

Theorem 2.1 implies that the grand coalition is formed immediately from the initial coalition structure π^0 consisting of single players when $\delta \leq \overline{\delta}(\pi^0)$. If this holds then it is also optimal to propose the grand coalition immediately for every $\pi \neq \{N\}$. Furthermore, let π be a coalition structure with $|\pi| = 2$, so π consists of two coalitions, say C_1 and C_2. Then

$$\overline{\delta}(\pi) = \min_{\{\pi'\leq\pi:|\pi'|\neq 1\}} \ \delta(\pi') = \delta(\pi) = \min_{S=\{C_1\},S=\{C_2\}} \ \delta_\pi(S) = 1,$$

because taking either $S = \{C_1\}$ or $S = \{C_2\}$ yields $\delta_\pi(S) = 1$. So, this implies that from a coalition structure with two coalitions the grand coalition is formed immediately for every value of $\delta \in [0,1)$ and, thus, Lemma 2.1 is a special case of Theorem 2.1. This is in accordance with the results for the bilateral alternating-offer model with random proposers in e.g. Muthoo (1999). Moreover, every coalition structure π always contains coarser coalition structures π' with two coalitions and, hence, $\overline{\delta}(\pi) \leq 1$.

Finally, we investigate the necessary and sufficient condition for the grand coalition to form immediately in all $\pi' \leq_p \pi^0$ for all $\delta \in [0,1)$, i.e., $\overline{\delta}(\pi^0) = 1$. Notice that $\overline{\delta}(\pi^0) = 1$ if and only if $\delta_\pi(S) \geq 1$ for every $\pi \neq \{N\}$ and every $S \subset \pi$. So, $\overline{\delta}(\pi^0) = 1$ if and only if for every $\pi \neq \{N\}$ and every $S \subset \pi$ it holds that

$$d_N(\{N\}) - d_{S^\cup}(\pi_S) - \sum_{B \in \pi \setminus S} d_B(\pi) \geq$$
$$\left(d_N(\{N\}) - \sum_{B \in \pi} d_B(\pi)\right) \sum_{B \in \pi \setminus S} p_B + p_{S^\cup}\left(d_N(\{N\}) - \sum_{B \in \pi_S} d_B(\pi_S)\right).$$

By rearranging terms this reduces to the inequality

$$d_{S^\cup}(\pi_S) + p_{S^\cup}\left(d_N(\{N\}) - \sum_{B \in \pi_S} d_B(\pi_S)\right) \leq \sum_{B \in S}\left[d_B(\pi) + p_B\left(d_N(\{N\}) - \sum_{B \in \pi} d_B(\pi)\right)\right]. \tag{2.9}$$

From above, we know that these inequalities only impose restrictions for coalition structures with three or more coalitions, i.e., $|\pi| \geq 3$. This yields the following corollary.

Corollary 2.1. *For every subgame at state π and every proposer in π it is optimal to propose the grand coalition if and only if for every π such that $|\pi| \geq 3$ and every $S \subset \pi$ inequality* (2.9) *holds.*

Proposition 4 of Gomes (2005) states that the grand coalition is formed immediately if the inequality (2.9) holds, i.e., is a sufficient condition. The corollary shows that Proposition 4 follows as a special case from Theorem 2.1 and, moreover, that it is a necessary and sufficient condition. Moreover, we also conclude that coalition structures with two coalitions can be neglected.

2.3 Internal Stability

The concept of stability is firstly introduced in d‘Aspremont et al. (1983) for the cartel collusive price. There is only one coalition with more than one player (i.e., the cartel) and the other players remain as a singleton. Internal stability of a cartel means that no player in the cartel has an incentive to leave it, i.e., a player who leaves the cartel gets a lower payoff compared with his current payoff in the coalition. External stability of a cartel means that no player outside the cartel has an incentive to join the cartel, i.e., if he joins the cartel, he gets a lower payoff in the cartel compared with his payoff under the “status quo”. If the conditions for internal and external stability are satisfied, the cartel is said to be stable. In d‘Aspremont et al. (1983), the whole analysis is based on the assumption that all players are identical. The application of stability to International Environmental

Agreements is largely hampered by the symmetry assumption because most countries are not identical. For instance, in an International Water Sharing Agreement, countries might differ in its catchment in a trans-boundary river basin and also the economic dependence on this river basin. Hence, the distribution of the benefits of an International Water Sharing Agreement is crucial for individual incentives to join the agreement and, hence, for the stability of the agreement. Weikard (2009) analyzes the stability of cartels in games with heterogeneous players and externalities. He introduces a class of sharing rules for coalition payoffs, called "optimal sharing rules", that stabilize all cartels that are possibly stable under some arbitrary sharing rule. Similar analyses can be found in Eyckmans and Finus (2004) and Weikard et al. (2006). Note that all these references rely on the assumption that there is only one coalition with more than one player among all players, thus only one cartel.

Like mentioned above, the concept of stability is relevant for the International Water Sharing Agreements since it identifies the "internal" incentives for all coalition members to cooperate. For instance, the four Lower Mekong Countries, i.e., Thailand, Laos, Cambodia and Vietnam, signed the 1995 Agreement for "sustainable development, utilization, conservation and management of the Mekong River Basin water and related resources". In reality, there is no legal body to enforce the agreement in trans-boundary river basins. It is paramount that all members in the agreement must have incentives to comply with the signed treaty. The essential idea is that the payoff of the coalition is sufficiently large such that it is able to satisfy each coalition member's "outside option". In this dissertation, each member's "outside option" is defined as his payoff if he unilaterally deviates from the coalition and remains as a singleton.

Let the collection of all possible partitions of $N = \{1, 2, \cdots, n\}$ be Π and the exogenous partition into m groups of agents be $\pi = \{P_1, P_2, \ldots, P_m\}$.[5] We call an element in π a group of agents. Note that $|P_1| + |P_2| + \cdots + |P_m| = n$, $m \leq n$, where $|P_j|$, $j = 1, \ldots, m$, is the number of agents in coalition P_j, $P_j \in \pi$. In case $m = n$, each agent is a singleton. In case $m = 1$, the grand coalition is formed. Let the payoff for each group of agents in partition π be $d(P_j|\pi)$, $P_j \in \pi$.[6] We need to assign a payoff to each agent $i \in P_j$, $P_j \in \pi$.

Definition 2.2. *An allocation rule γ is a function $v : \Pi \to \mathbb{R}^N$ that assigns a payoff $v_i(\pi)$ to any $i \in N$, for all $\pi \in \Pi$.*

The group efficiency condition for the allocation rule γ is imposed, i.e., $\sum_{i \in P_j} v_i(\pi)$ $= d(P_j|\pi)$ for any $P_j \in \pi, \pi \in \Pi$.

The next question is whether the allocation rule γ makes the partition π internally stable. By saying this, we mean that there is no beneficial unilateral deviation for a player to be a singleton from the current partition π. For instance, consider the deviation of one agent leaving his group of agents in partition π. Let $i \in P_j$ be the deviating agent that leaves $P_j \in \pi$. After agent i leaves P_j, the new partition will be $\pi^i = \{P_1, P_2, \ldots, P_j \setminus \{i\}, \{i\}, \ldots, P_m\}$. Note that the new partition is finer and has $m + 1$ groups of agents including at least one singleton $\{i\}$, $i \in N$. The payoff for the deviating agent i under π^i serves as his "outside option" in the current partition π.

[5]Note that in Section 2.2, we use the term coalition structure instead of partition.

[6]We will discuss more details about how to calculate $d(P_j|\pi)$ in Chapter 5.

Definition 2.3. *Under allocation rule γ, partition π is internally stable if $v_i(\pi) \geq d(\{i\}|\pi^i)$ for all $i \in N$.*

Note that under group efficiency condition, for any allocation rule, the partition $\pi = \{\{i\}, i \in N\}$ is internally stable.

Proposition 2.2. *There exists an allocation rule γ to make partition π internally stable if and only if $d(P_j|\pi) \geq \sum_{i \in P_j} d(\{i\}|\pi^i)$, for all $P_j \in \pi$.*

The above proposition is straightforward. Note that it could be well possible that $d(P_j|\pi) < \sum_{i \in P_j} d(\{i\}|\pi^i)$, $P_j \in \pi$, see, e.g., Theorem 2 and Example 1 in Ambec and Ehlers (2008). In this scenario, the payoff of the coalition under π is insufficient to satisfy each agent's "outside option" in terms of unilateral deviation from the current coalition. In Chapter 5, we will consider a scenario with only one partial coalition with connected agents (of more than one agents) in the International Water Sharing Agreement and investigate the stability of the coalition. An empirical study to the Mekong River Basin is also presented in Chapter 5.

2.4 Appendix with proofs of Chapter 2

Proof of Lemma 2.1

Consider the proposal $\left(S, t^S(\pi)\right) = (\pi, t^\pi(\pi))$ for coalition C, then in (2.7) the first term on the right-hand side becomes

$$
\begin{aligned}
& (1-\delta)d_N(\{N\}) + \delta v_N(\{N\}) - \sum_{B \in \pi} \left[(1-\delta)d_B(\pi) + \delta v_B(\pi)\right] \\
= \; & (1-\delta)\left[d_N(\{N\}) - \sum_{B \in \pi} d_B(\pi)\right] + \delta \left[v_N(\{N\}) - \sum_{B \in \pi} v_B(\pi)\right] > 0,
\end{aligned}
$$

because the first term is positive by Assumption 2.1 and the second term is nonnegative. So, the maximum has to be positive, which rules out that proposing $\left(S, t^S(\pi)\right) = \left(\{C\}, t^{\{C\}}(\pi)\right)$ is optimal. QED.

Proof of Lemma 2.2

If $\sigma_B(\pi|\pi) = 1$ is optimal for every $B \in \pi$, $\pi \neq \{N\}$, then $\sum_{\substack{S \subset \pi \\ S \ni B, C}} \sigma_B(S|\pi) = 1$ and $\sum_{\substack{S \subset \pi \\ S \ni B, C \notin S}} \sigma_B(S|\pi) = 0$. So, Equation (2.7) degenerates to

$$
\begin{aligned}
v_C(\pi) = \; & p_C \left[(1-\delta)d_N(\{N\}) + \delta v_N(\{N\}) - \sum_{B \in \pi} \left[(1-\delta)d_B(\pi) + \delta v_B(\pi)\right]\right] \\
& + \left[(1-\delta)d_C(\pi) + \delta v_C(\pi)\right].
\end{aligned}
$$

Summing over all $B \in \pi$ and substituting $v_N(\{N\}) = d_N(\{N\})$ yields $\sum_{B\in\pi} v_B(\pi) = d_N(\{N\})$. Then the equation for $v_C(\pi)$ can be rewritten into

$$(1-\delta)v_C(\pi) = (1-\delta)d_C(\pi)+p_C\left[(1-\delta)(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)) + \delta\left[d_N(\{N\}) - \sum_{B\in\pi} v_B(\pi)\right]\right]$$

and we immediately obtain

$$v_C(\pi) = d_C(\pi) + p_C\left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right).$$

QED.

Proof of Theorem 2.1

Only if: The equilibrium conditions such that every proposer in the subgame at state π proposes the grand coalition impose that

$$\begin{aligned}&(1-\delta)\, d_N(\{N\}) + \delta v_N\left(\{N\}\right) - \sum_{B\in\pi}\left[(1-\delta)\, d_B\left(\pi\right) + \delta v_B\left(\pi\right)\right]\\ &\geq (1-\delta)\, d_{S^\cup}(\pi_S) + \delta v_{S^\cup}(\pi_S) - \sum_{B\in S}\left[(1-\delta)\, d_B\left(\pi\right) + \delta v_B\left(\pi\right)\right],\end{aligned}$$

for all $S \subset \pi$. Rewriting several times yields

$$\begin{aligned}&\delta\left[v_{S^\cup}(\pi_S) - d_{S^\cup}(\pi_S) + \left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right)\sum_{B\in\pi\setminus S} p_B\right]\\ &\leq d_N(\{N\}) - d_{S^\cup}(\pi_S) - \sum_{B\in\pi\setminus S} d_B(\pi).\end{aligned}$$

Notice that we consider strategies in which every proposer in every subgame $\pi' \leq_p \pi$ proposes the grand coalition. Then according to Lemma 2.2 we have

$$v_{S^\cup}(\pi_S) = d_{S^\cup}(\pi_S) + p_{S^\cup}\left(d_N(\{N\}) - \sum_{B\in\pi_S} d_B(\pi_S)\right).$$

By further substitution, we get

$$\begin{aligned}&\delta\left[p_{S^\cup}\left(d_N(\{N\}) - \sum_{B\in\pi_S} d_B(\pi_S)\right) + \left(d_N(\{N\}) - \sum_{B\in\pi} d_B(\pi)\right)\sum_{B\in\pi\setminus S} p_B\right]\\ &\leq d_N(\{N\}) - d_{S^\cup}(\pi_S) - \sum_{B\in\pi\setminus S} d_B(\pi).\end{aligned}$$

Since the grand coalition is efficient, the term in the square bracket on the left-hand side is positive. Then, $\delta \leq \delta_\pi(S)$ follows immediately. By taking the minimum over all $S^\cup$

we obtain $\delta \leq \delta(\pi)$ for every coalition structure π. Finally, by taking the minimum over π and coalitions structures coarser than π, i.e., $\{\pi' \leq_p \pi : |\pi'| \neq 1\}$, we get the desired result.
If: When $\delta \leq \overline{\delta}(\pi)$, we prove the result recursively. Let π be a coalition structure with $|\pi| = 2$, so π consists of two coalitions, say C_1 and C_2. Then

$$\overline{\delta}(\pi) = \min_{\{\pi' \leq_p \pi : |\pi'| \neq 1\}} \delta(\pi') = \delta(\pi) = \min_{S=\{C_1\}, S=\{C_2\}} \delta_\pi(S).$$

Taking either $S = \{C_1\}$ or $S = \{C_2\}$ yields $\delta_\pi(S) = 1$, implying that from a coalition structure with two coalitions the grand coalition is formed immediately for every value of δ. Now consider $|\pi| = 3$, when $\delta \leq \overline{\delta}(\pi)$, it can be easily shown that for every proposer in π, it is optimal to propose the grand coalition immediately. For $\pi = \pi^0$ and $|\pi| > 3$, we can deduce backwards in a similar way. QED.